Gravitational Waves and Coalescing Black Holes

Thibault Damour

Institut des Hautes Etudes Scientifiques

Department of Mathematics
National University of Singapore
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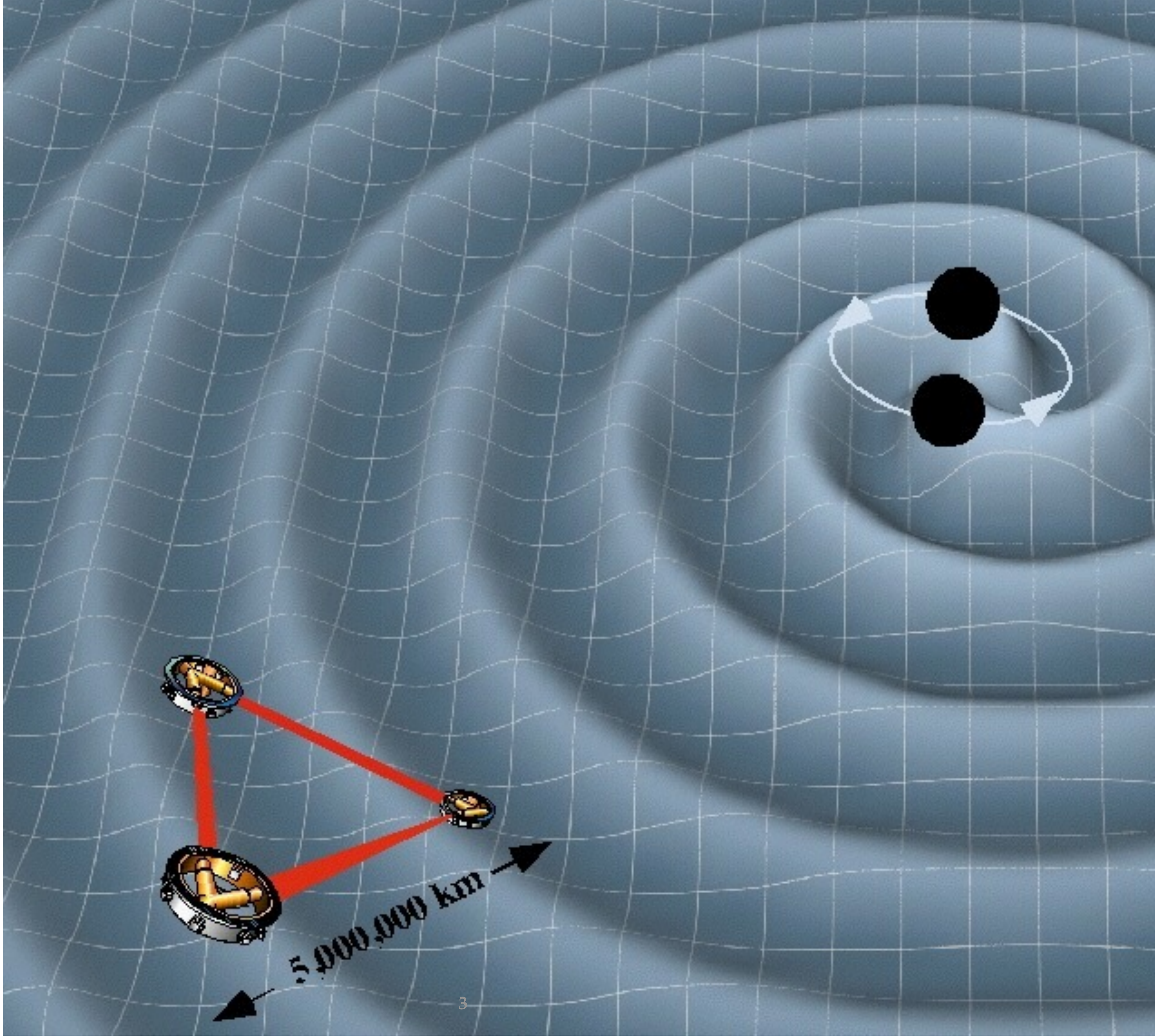
THE TWO LIGO INTERFEROMETRIC GRAVITATIONAL WAVE DETECTORS

LIGO Hanford

LIGO Livingston



$$m_1 = 36^{+5}_{-4} M_\odot$$

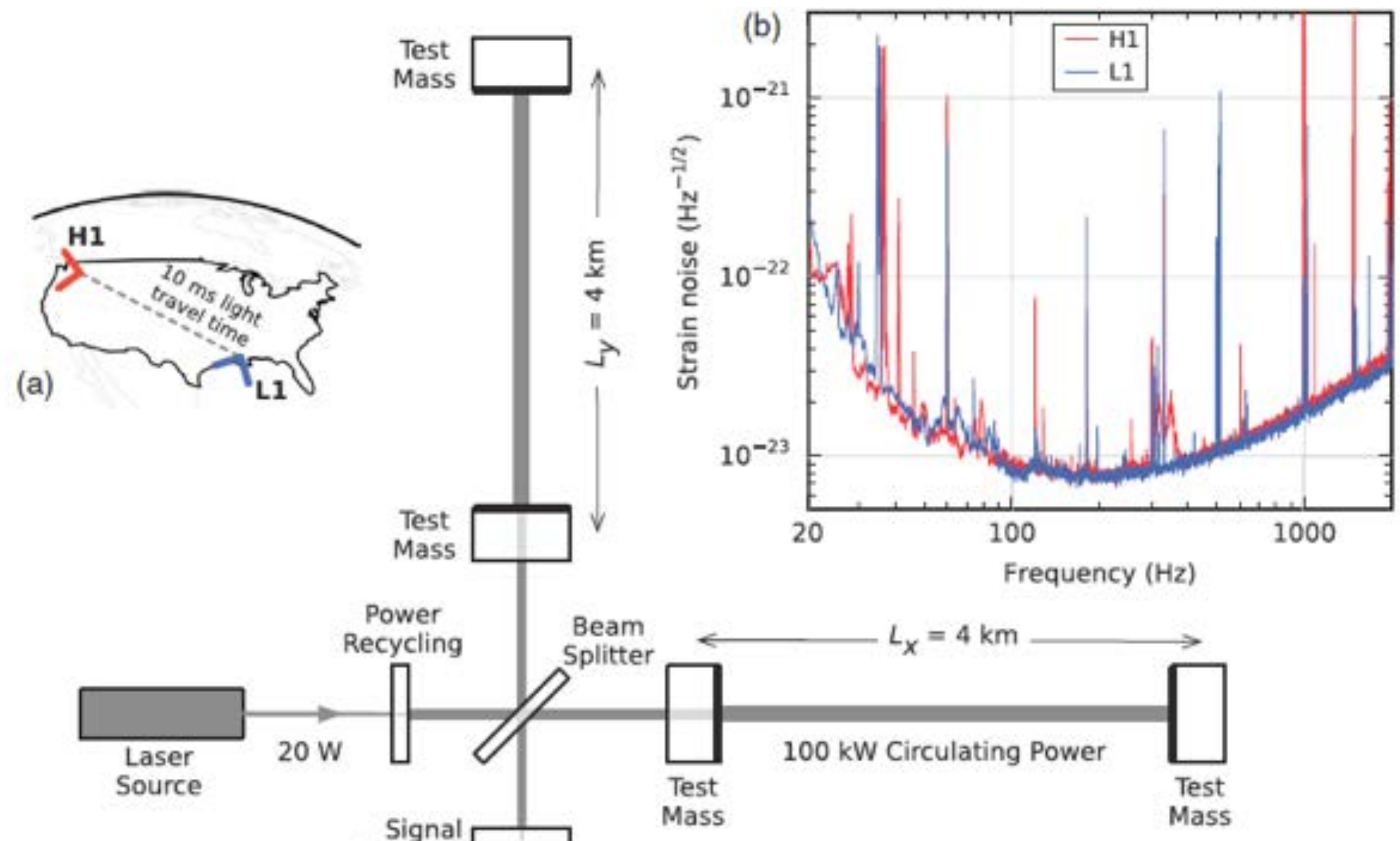
$$m_2 = 29^{+4}_{-4} M_\odot$$

$$\chi_{\text{eff}} = -0.06^{+0.17}_{-0.18}$$

$$D_{\text{L}} = 410^{+160}_{-180}\text{Mpc}$$

LASER INTERFEROMETER GW DETECTORS

[Weber]
1962 Gertsenshtein-Pustovoit
1970 Forward
1972 Weiss
1972 Levine-Hall, Levine-Stebbins
1979 Billing H, Maischberger K, Ruediger A, Schilling R, Schnupp L, Winkler, W;
1979 Drever, Hough, Pugh, Edelstein, Ward, Ford, Robertson
1981 Drever et al
1982 Brillet, Man, Vinet
1980 Giazzoto
1988 Shoemaker et al
1988 Meers
....
2016 Abbott et al. LIGO-Virgo collab.



Abbott et al. PRL 11 Feb 2016

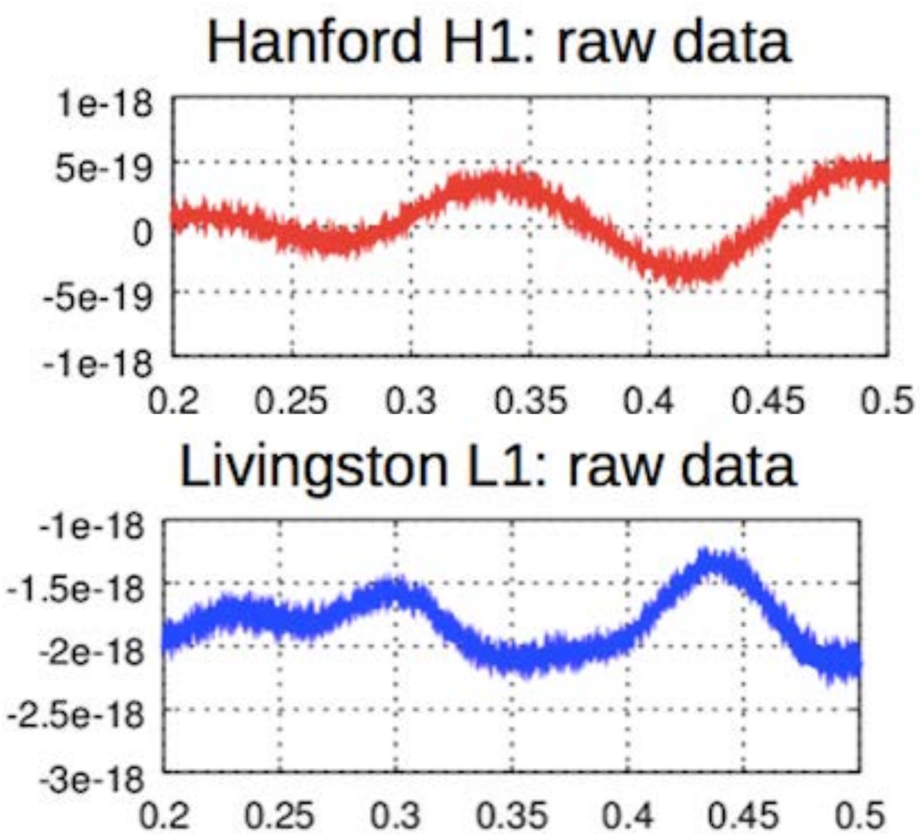
LIGO Hanford

LIGO Livingston

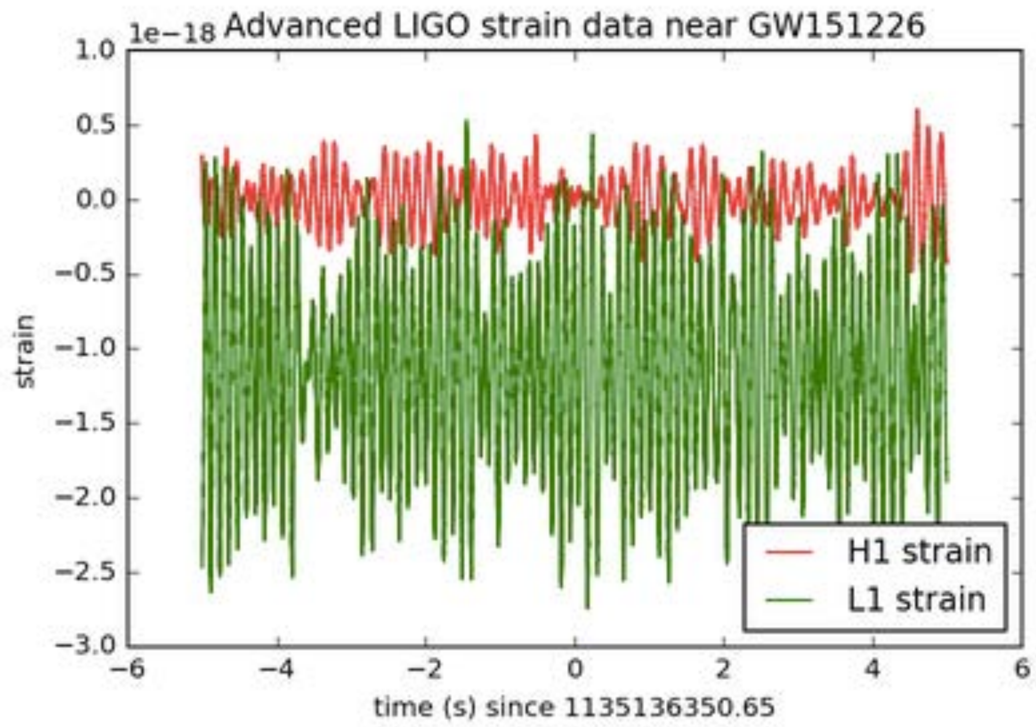
Virgo (IT)

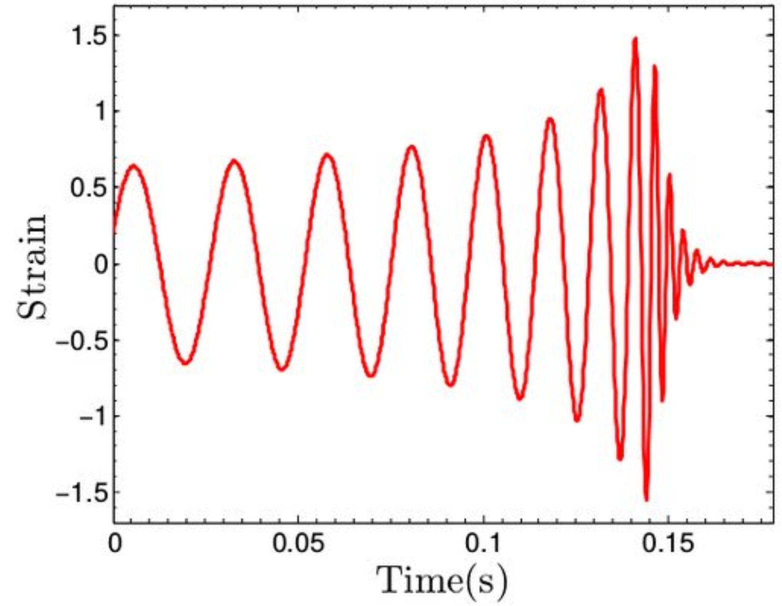
GW150914 and GW151226: incredibly small signals lost in the broad-band noise

GW150914, from Chassande-Mottin 5 April 2016



GW151226 from LIGO open data





$$h_{GW}^{\rm max} \sim 10^{-21} \sim 10^{-3}\, h_{LIGO}^{\rm broadband}$$

$$\delta L/L = 10^{-21} \rightarrow \delta L \sim 10^{-9}\,{\rm atom}\,!$$

Matched Filtering

$$\langle output | h_{\rm template} \rangle = \int \frac{df}{S_n(f)} o(f) h^*_{\rm template}(f)$$

MATCHED FILTERING SEARCH AND DATA ANALYSIS

Precomputed bank of 250 000 templates for inspiralling and coalescing BBH GW waveforms: m1, m2, chi1=S1/m1^2, chi2=S2/m2^2

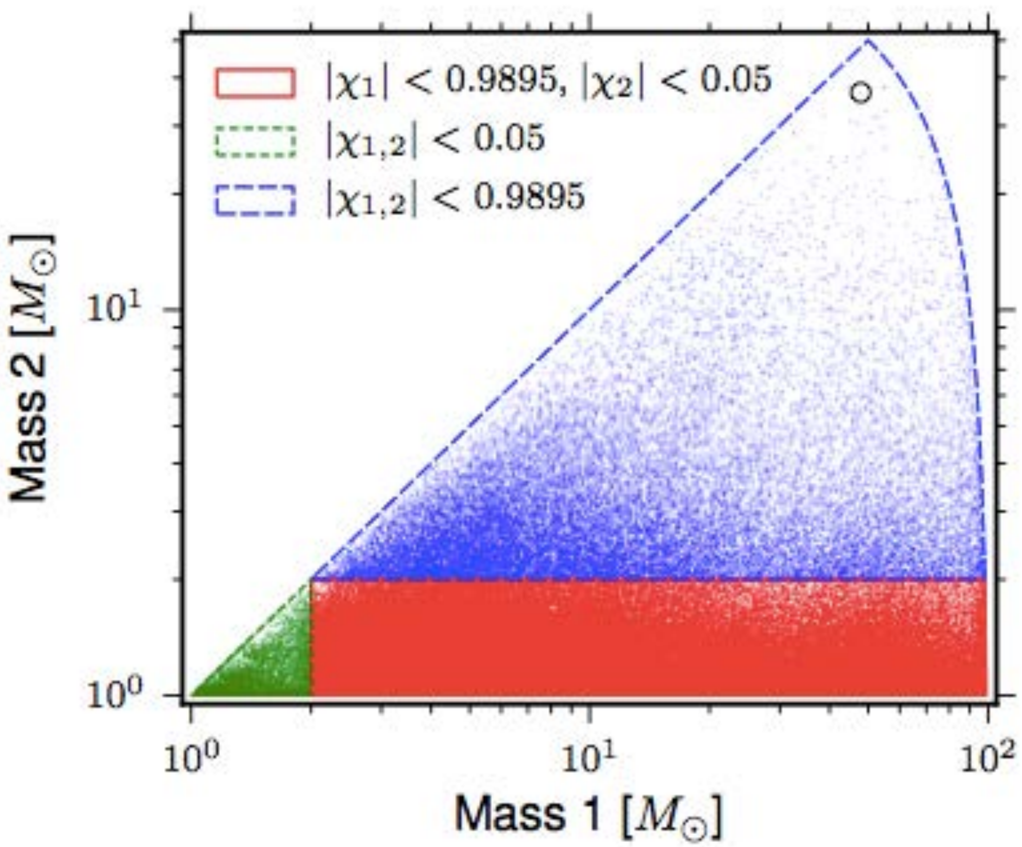
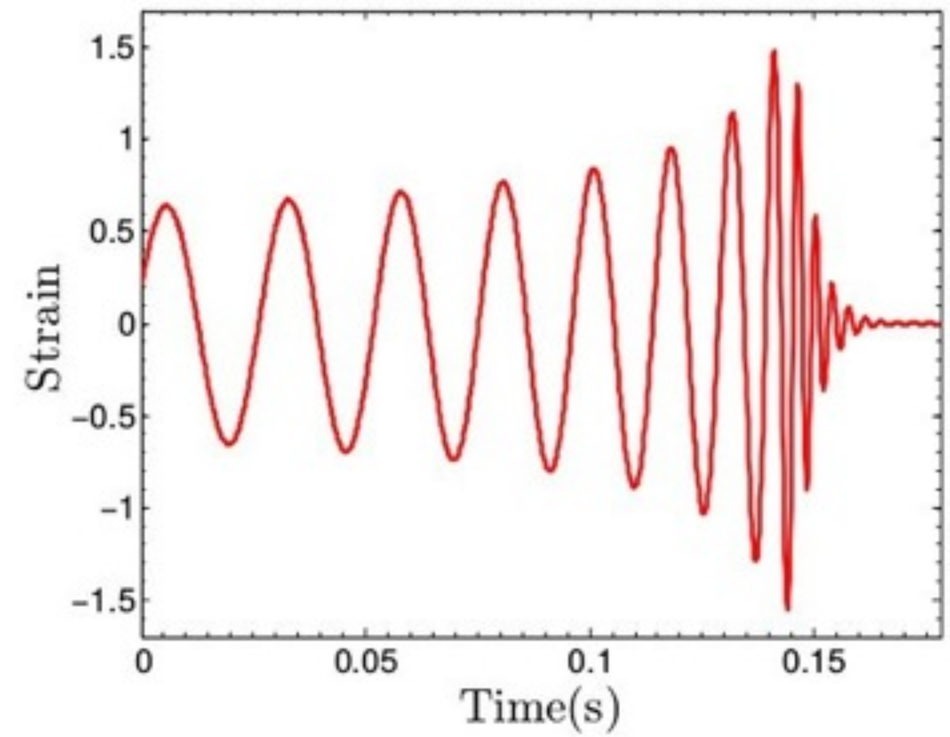


FIG. 1. The four-dimensional search parameter space covered by the template bank shown projected into the component-mass plane, using the convention $m_1 > m_2$. The lines bound mass regions with different limits on the dimensionless aligned-spin parameters χ_1 and χ_2. Each point indicates the position of a template in the bank. The circle highlights the template that best matches GW150914. This does not coincide with the best-fit parameters due to the discrete nature of the template bank.

Search template bank made of analytical EOB[NR] templates (Buonanno-Damour 99…, Taracchini et al. 14) in ROM form; [post-computed NR waveform for GW151226 took three months and 70 000 CPU hours !]
Several data analysis pipelines: PyCBC, GstLAL (+ Phenom[EOB+NR])



November 25 1915, January 13, 1916, June 1916, January 1918

Die Feldgleichungen der Gravitation.

Von A. Einstein.

In zwei vor kurzem erschienenen Mitteilungen[1] habe ich gezeigt, wie man zu Feldgleichungen der Gravitation gelangen kann, die dem Postulat allgemeiner Relativität entsprechen, d. h. die in ihrer allgemeinen Fassung beliebigen Substitutionen der Raumzeitvariabeln gegenüber kovariant sind.

Der Entwicklungsgang war dabei folgender. Zunächst fand ich Gleichungen, welche die Newtonsche Theorie als Näherung enthalten und beliebigen Substitutionen von der Determinante 1 gegenüber kovariant waren. Hierauf fand ich, daß diesen Gleichungen allgemein kovariante entsprechen, falls der Skalar des Energietensors der »Materie« verschwindet. Das Koordinatensystem war dann nach der einfachen Regel zu spezialisieren, daß $\sqrt{-g}$ zu 1 gemacht wird, wodurch die Gleichungen der Theorie eine eminente Vereinfachung erfahren. Dabei mußte aber, wie erwähnt, die Hypothese eingeführt werden, daß der Skalar des Energietensors der Materie verschwinde.

Neuerdings finde ich nun, daß man ohne Hypothese über den Energietensor der Materie auskommen kann, wenn man den Energietensor der Materie in etwas anderer Weise in die Feldgleichungen einsetzt, als dies in meinen beiden früheren Mitteilungen geschehen ist. Die Feldgleichungen für das Vakuum, auf welche ich die Erklärung der Perihelbewegung des Merkur gegründet habe, bleiben von dieser Modifikation unberührt. Ich gebe hier nochmals die ganze Betrachtung, damit der Leser nicht genötigt ist, die früheren Mitteilungen unausgesetzt heranzuziehen. [2]

Aus der bekannten Riemannschen Kovariante vierten Ranges leitet man folgende Kovariante zweiten Ranges ab:

$$G_{im} = R_{im} + S_{im} \qquad (1)$$

$$R_{im} = -\sum_l \frac{\partial \begin{Bmatrix} im \\ l \end{Bmatrix}}{\partial x_l} + \sum_{l\rho} \begin{Bmatrix} il \\ \rho \end{Bmatrix} \begin{Bmatrix} m\rho \\ l \end{Bmatrix} \qquad (1a)$$

$$S_{im} = \sum_l \frac{\partial \begin{Bmatrix} il \\ l \end{Bmatrix}}{\partial x_m} - \sum_{l\rho} \begin{Bmatrix} im \\ \rho \end{Bmatrix} \begin{Bmatrix} \rho l \\ l \end{Bmatrix} \qquad (1b)$$

[1] Sitzungsber. XLIV, S. 778 und XLVI, S. 799, 1915. [1]

Sitzber. Deut. Akad. Wiss. (1916)

Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie.

Von K. Schwarzschild.

(Vorgelegt am 13. Januar 1916 [s. oben S. 42].)

§ 1. Hr. Einstein hat in seiner Arbeit über die Perihelbewegung des Merkur (s. Sitzungsberichte vom 18. November 1915) folgendes Problem gestellt:

Ein Punkt bewege sich gemäß der Forderung

$$\left.\begin{aligned} &\delta \int ds = 0, \\ \text{wobei} \quad &ds = \sqrt{\sum g_{\mu\nu} dx_\mu dx_\nu} \quad \mu,\nu = 1,2,3,4 \end{aligned}\right\} \qquad (1)$$

ist, $g_{\mu\nu}$ Funktionen der Variabeln x bedeuten und bei der Variation am Anfang und Ende des Integrationswegs die Variablen x festzuhalten sind. Der Punkt bewege sich also, kurz gesagt, auf einer geodätischen Linie in der durch das Linienelement ds charakterisierten Mannigfaltigkeit.

Die Ausführung der Variation ergibt die Bewegungsgleichungen des Punktes

$$\frac{d^2 x_\alpha}{ds^2} = \sum_{\mu,\nu} \Gamma^\alpha_{\mu\nu} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds}, \quad \alpha,\beta = 1,2,3,4 \qquad (2)$$

wobei

$$\Gamma^\alpha_{\mu\nu} = -\frac{1}{2} \sum_\beta g^{\alpha\beta} \left(\frac{\partial g_{\mu\beta}}{\partial x_\nu} + \frac{\partial g_{\nu\beta}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\beta} \right) \qquad (3)$$

ist und $g^{\alpha\beta}$ die zu $g_{\alpha\beta}$ koordinierte und normierte Subdeterminante in der Determinante $|g_{\mu\nu}|$ bedeutet.

Dies ist nun nach der Einsteinschen Theorie dann die Bewegung eines masselosen Punktes in dem Gravitationsfeld einer im Punkt $x_1 = x_2 = x_3 = 0$ befindlichen Masse, wenn die »Komponenten des Gravitationsfeldes« Γ überall, mit Ausnahme des Punktes $x_1 = x_2 = x_3 = 0$, den »Feldgleichungen«

Über Gravitationswellen.

Von A. Einstein.

(Vorgelegt am 31. Januar 1918 [s. oben S. 79].)

Die wichtige Frage, wie die Ausbreitung der Gravitationsfelder erfolgt, ist schon vor anderthalb Jahren in einer Akademiearbeit von mir behandelt worden[1]. Da aber meine damalige Darstellung des Gegenstandes nicht genügend durchsichtig und außerdem durch einen bedauerlichen Rechenfehler verunstaltet ist, muß ich hier nochmals auf die Angelegenheit zurückkommen.

Wie damals beschränke ich mich auch hier auf den Fall, daß das betrachtete zeiträumliche Kontinuum sich von einem »galileischen« nur sehr wenig unterscheidet. Um für alle Indizes

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \qquad (1)$$

setzen zu können, wählen wir, wie es in der speziellen Relativitätstheorie üblich ist, die Zeitvariable x_4 rein imaginär, indem wir

$$x_4 = it$$

setzen, wobei t die »Lichtzeit« bedeutet. In (1) ist $\delta_{\mu\nu} = 1$ bzw. $\delta_{\mu\nu} = 0$, je nachdem $\mu = \nu$ oder $\mu \neq \nu$ ist. Die $\gamma_{\mu\nu}$ sind gegen 1 kleine Größen, welche die Abweichung des Kontinuums vom feldfreien darstellen; sie bilden einen Tensor vom zweiten Range gegenüber Lorentz-Transformationen.

§ 1. Lösung der Näherungsgleichungen des Gravitationsfeldes durch retardierte Potentiale.

Wir gehen aus von den für ein beliebiges Koordinatensystem gültigen[2] Feldgleichungen

$$-\sum_\alpha \frac{\partial}{\partial x_\alpha} \begin{Bmatrix} \mu\nu \\ \alpha \end{Bmatrix} + \sum_\alpha \frac{\partial}{\partial x_\nu} \begin{Bmatrix} \mu\alpha \\ \alpha \end{Bmatrix} + \sum_{\alpha\beta} \begin{Bmatrix} \mu\alpha \\ \beta \end{Bmatrix} \begin{Bmatrix} \nu\beta \\ \alpha \end{Bmatrix} - \sum_{\alpha\beta} \begin{Bmatrix} \mu\nu \\ \alpha \end{Bmatrix} \begin{Bmatrix} \alpha\beta \\ \beta \end{Bmatrix} = -\varkappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \qquad (2)$$

[1] Diese Sitzungsber. 1916, S. 688ff.

[2] Von der Einführung des »λ-Gliedes« (vgl. diese Sitzungsber. 1917, S. 142) ist dabei Abstand genommen.

Basics of Gravitational Waves

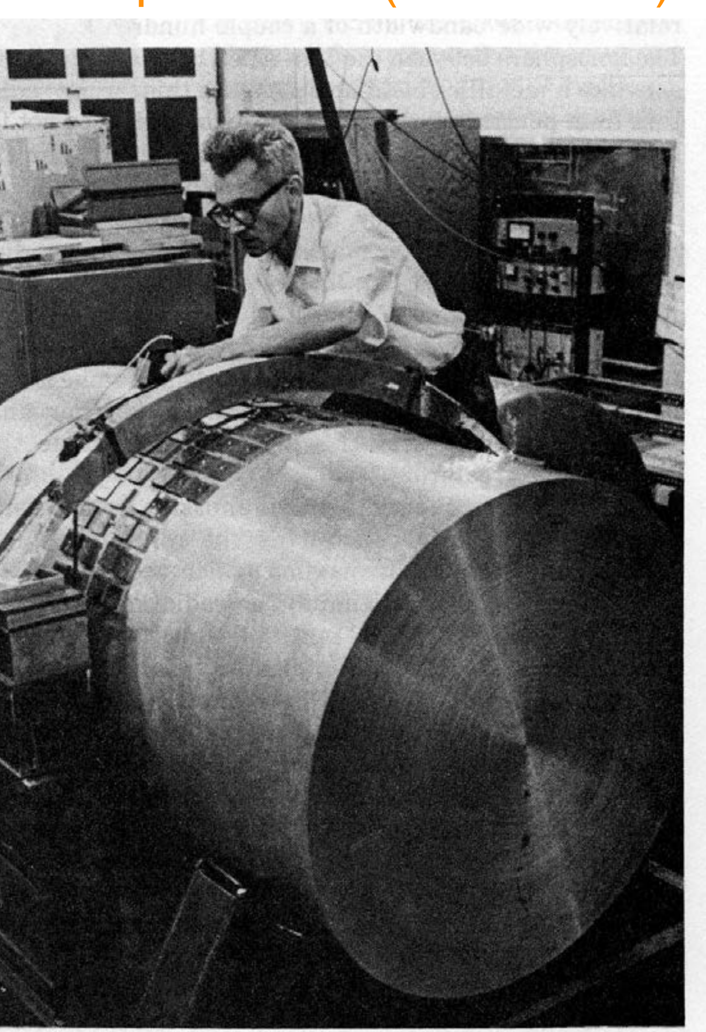
In linearized GR (Einstein 1916,1918):

$$\Box h_{\mu\nu} + \partial_\mu H_\nu + \partial_\nu H_\mu = -16\pi G(T_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}T)$$

$$H_\mu = \frac{1}{2}\partial_\mu h - \partial^\nu h_{\mu\nu} \qquad \partial^\nu T_{\mu\nu} = 0$$

$$ds^2 = -c^2 dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j$$

Joseph Weber (1919-2000)

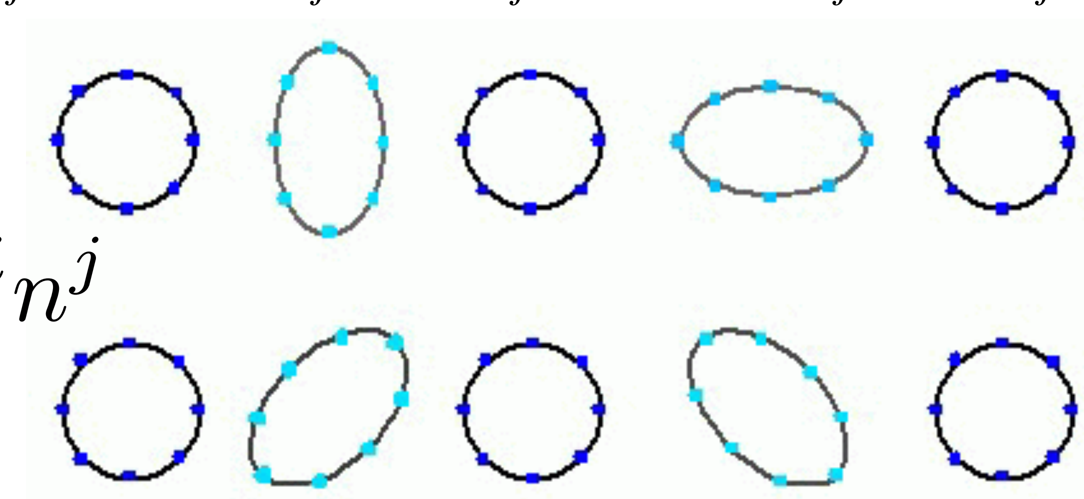


$$h_{ij} \simeq \frac{2G}{c^4 r}\ddot{Q}^{TT}_{ij}(t - r/c)$$

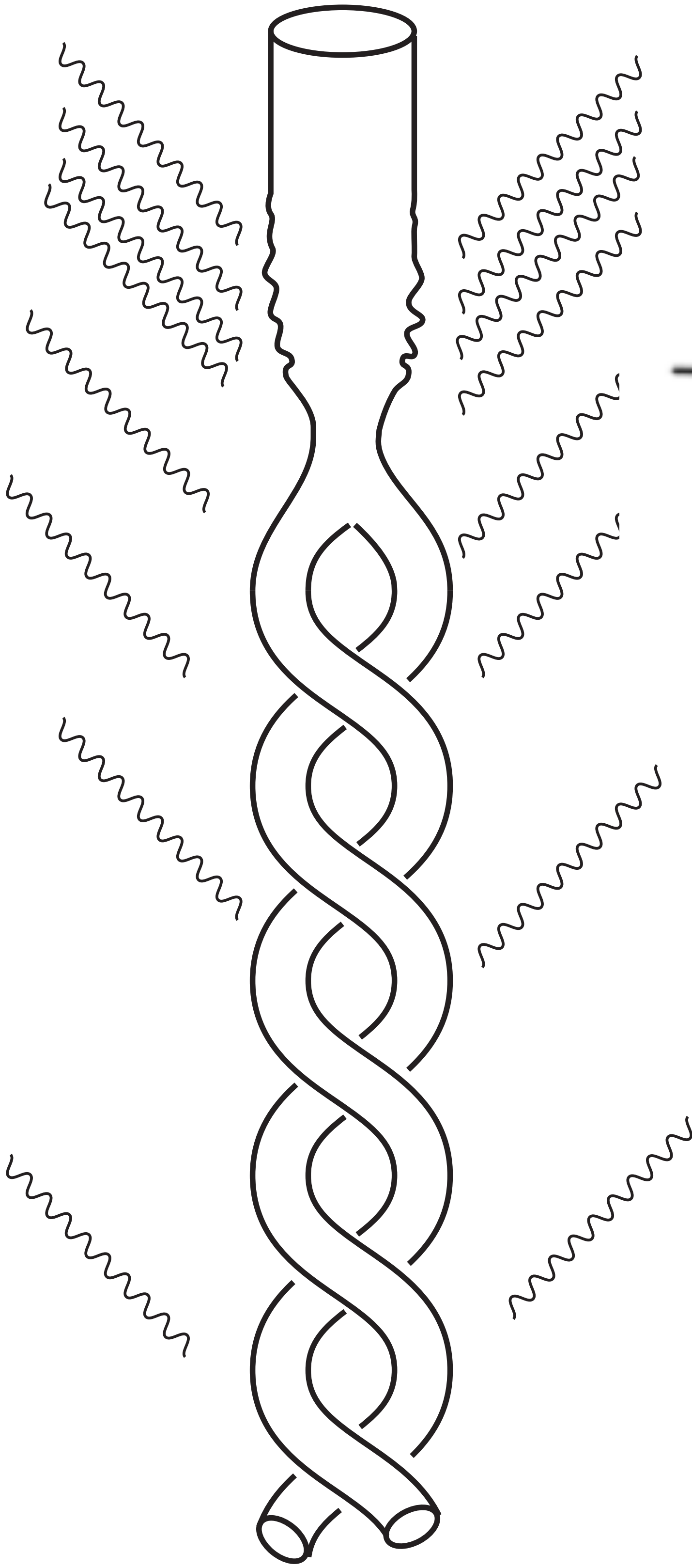
Two Transverse-Traceless (TT) tensor polarizations propagating at v=c

$$g_{ij} = \delta_{ij} + h_{ij} \quad h_{ij} = h_+(x_i x_j - y_i y_j) + h_\times(x_i y_j + y_i x_j)$$

$$\frac{\delta L}{L} = \frac{1}{2}h_{ij}n^i n^j$$



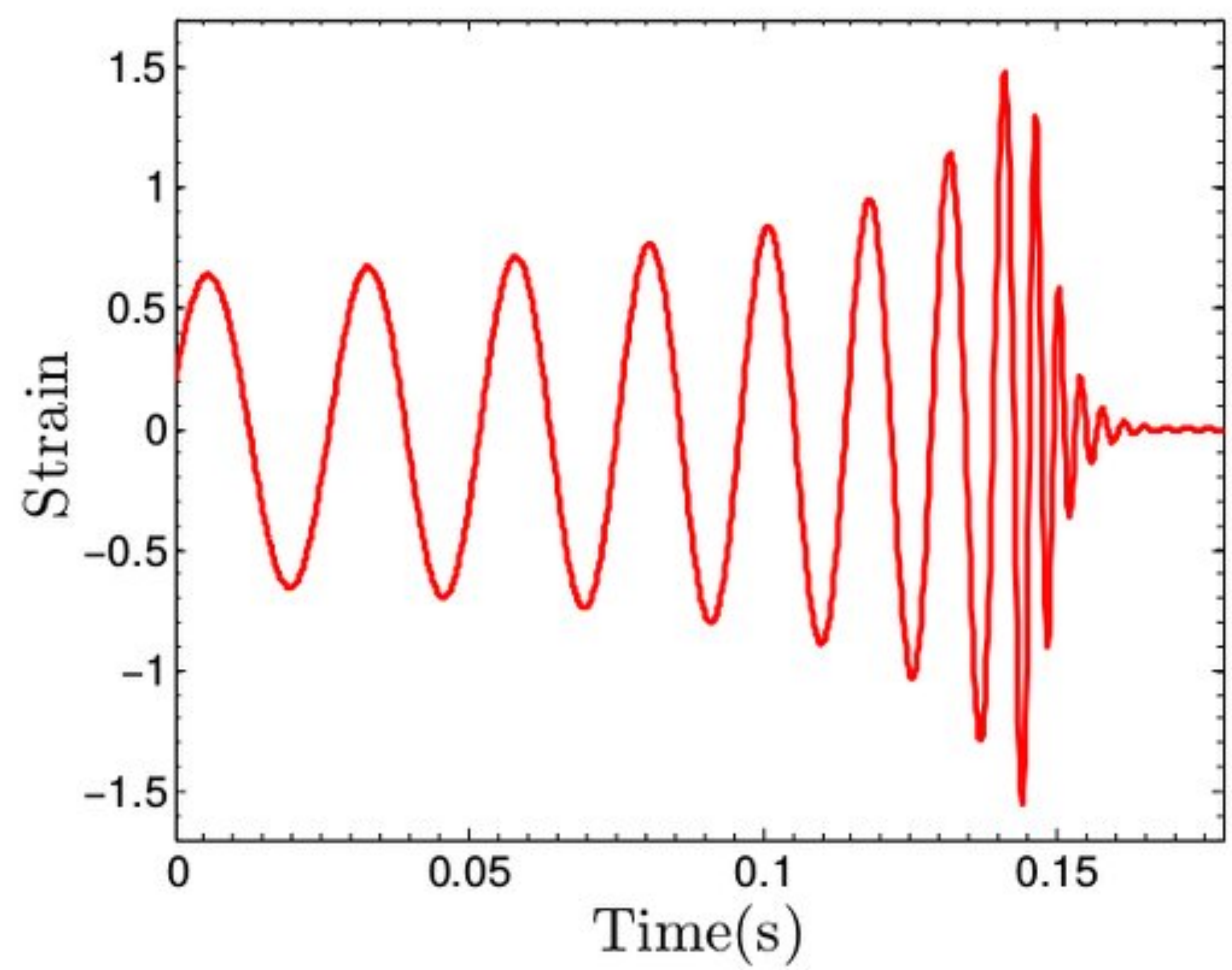
$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad R_{\mu\nu} = 0$$



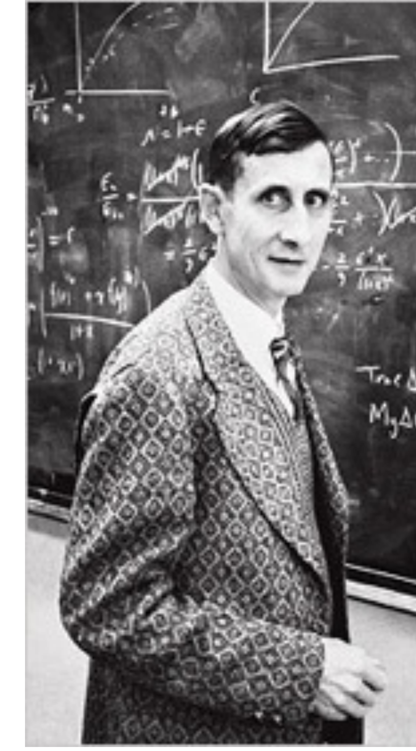
$$ds^2 = g_{\mu\nu}(x^\lambda)\, dx^\mu dx^\nu$$

$$-g^{\mu\nu} g_{\alpha\beta,\,\mu\nu} + g^{\mu\nu} g^{\rho\sigma} (g_{\alpha\mu,\,\rho} g_{\beta\nu,\,\sigma} - g_{\alpha\mu,\,\rho} g_{\beta\sigma,\,\nu} + g_{\alpha\mu,\,\rho} g_{\nu\sigma,\,\beta} + g_{\beta\mu,\,\rho} g_{\nu\sigma,\,\alpha} - \frac{1}{2} g_{\mu\rho,\,\alpha} g_{\nu\sigma,\,\beta})$$

$$= 0$$

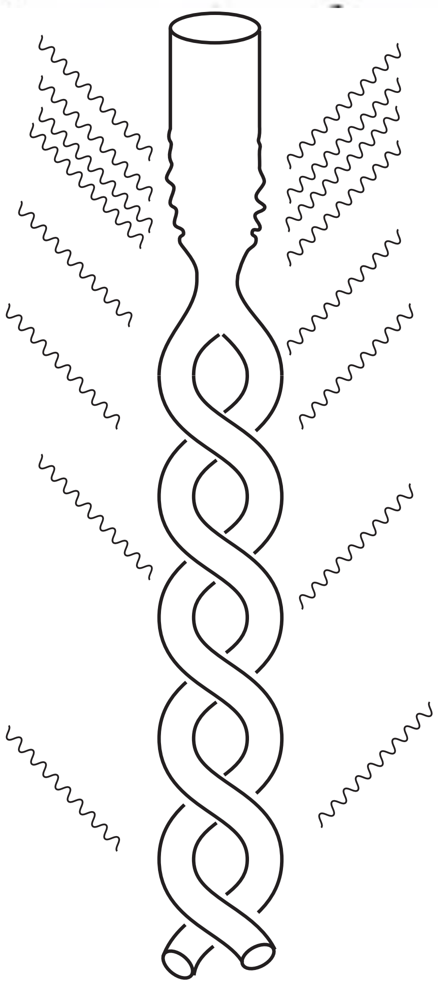


Pioneering the GWs from coalescing compact binaries



Freeman Dyson 1963, using Einstein 1918 + Landau-Lifshitz 1951 (+ Peters '64) first vision of an intense GW flash from coalescing binary NS

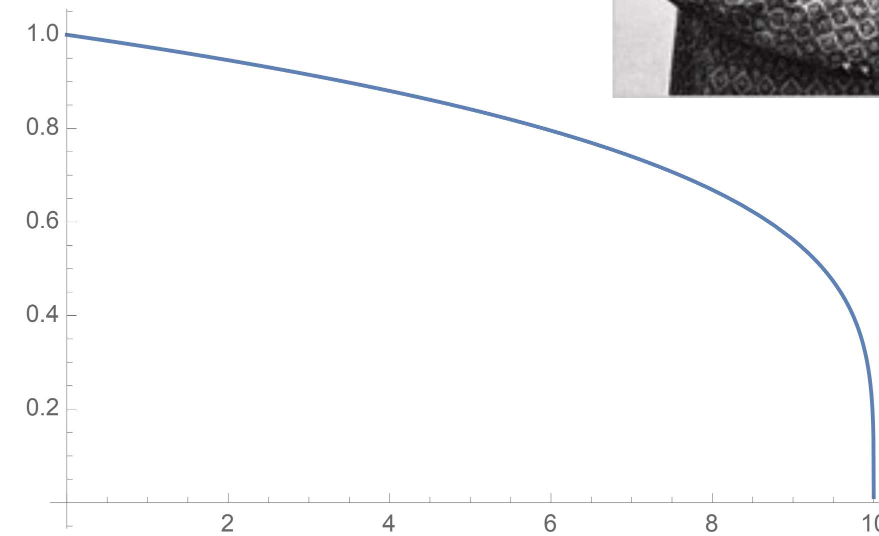
> but the final end will be the same. According to (11), the loss of energy by gravitational radiation will bring the two stars closer with ever-increasing speed, until in the last second of their lives they plunge together and release a gravitational flash at a frequency of about 200 cycles and of unimaginable intensity.

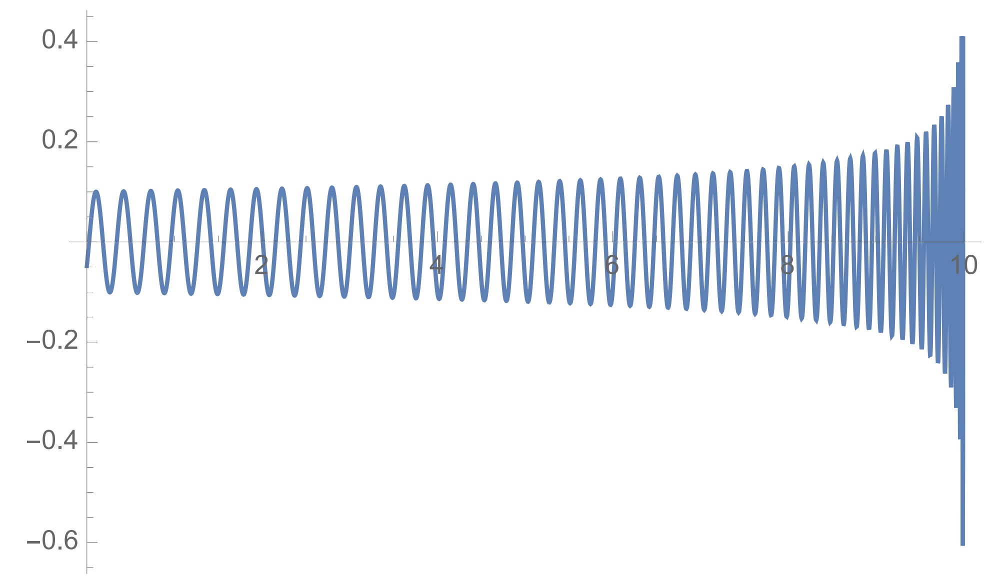


$$E = -\frac{G\, m_1\, m_2}{2r}$$

$$\frac{d}{dt} E = -F$$

$$F = \frac{32\, G^4}{5\, c^5} \frac{m_1^2\, m_2^2 (m_1 + m_2)}{r^5}$$

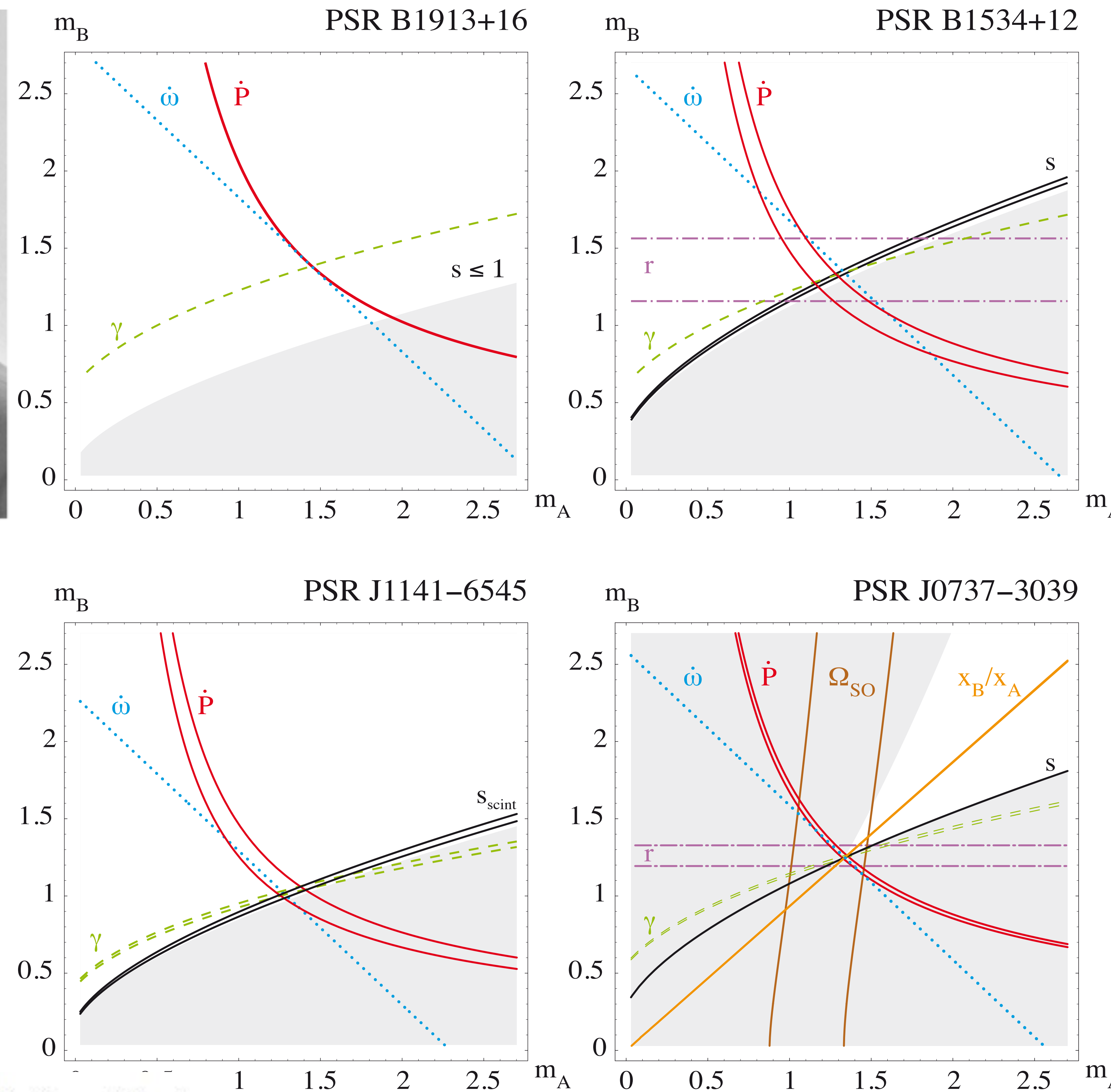


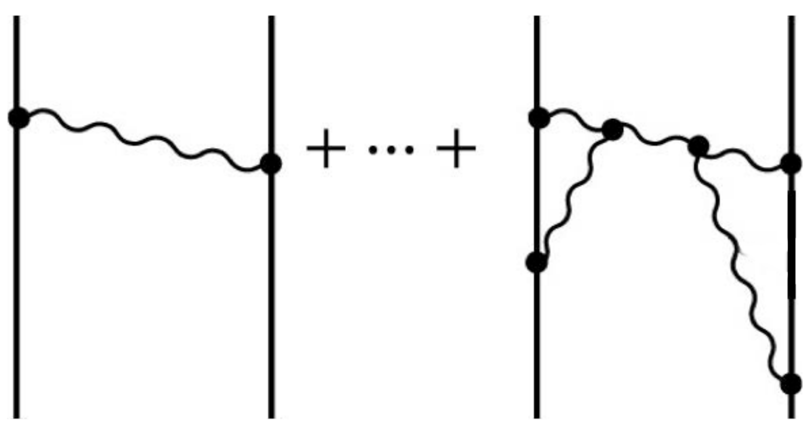


Challenge: describe the intense flash of GWs emitted by the last orbits and the merger of a binary BH, when v~c and r~GM/c^2

Binary pulsars: proof of radiative and strong-field gravity + existence of coalescing NS binaries

Russell Hulse Joseph Taylor

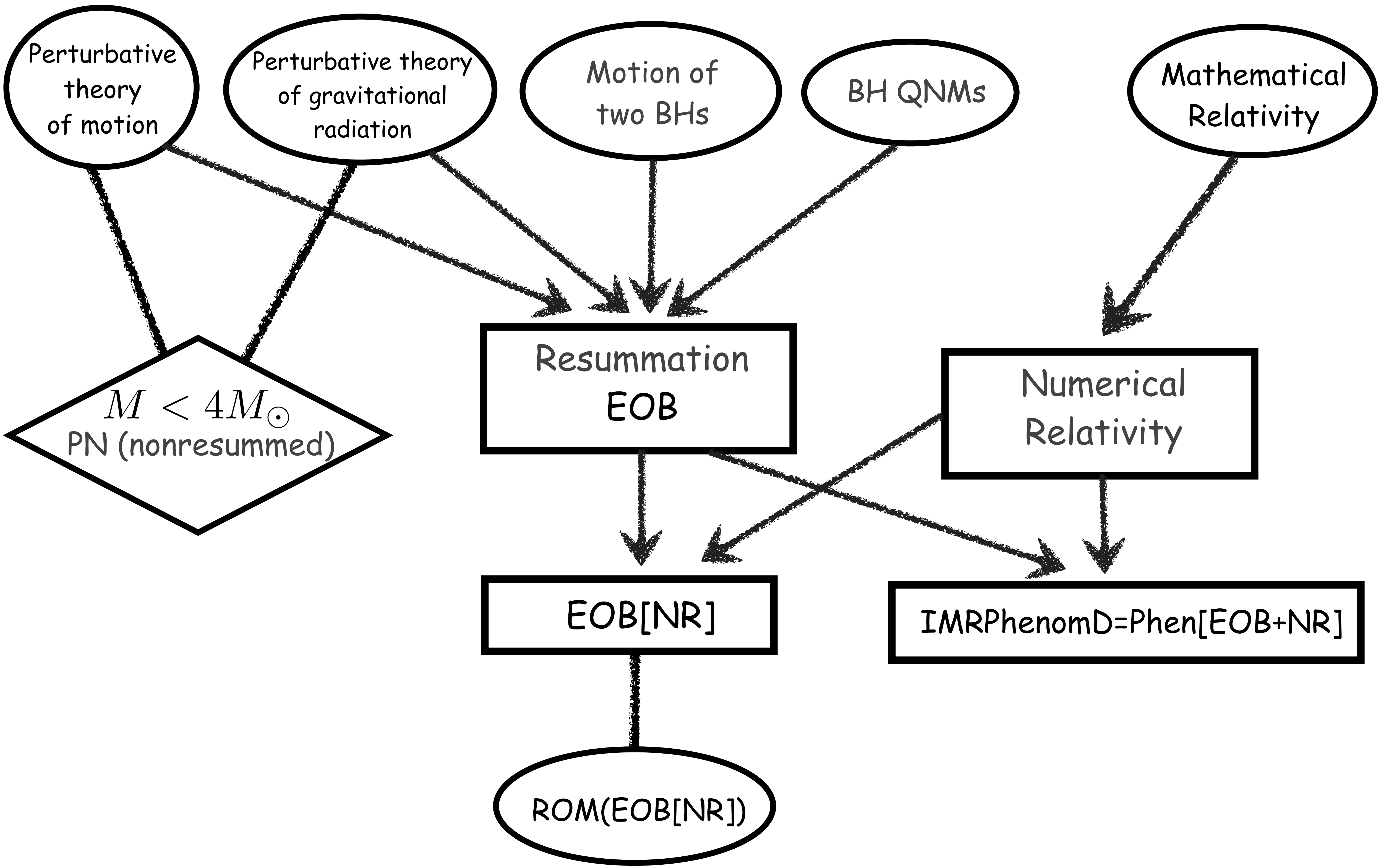


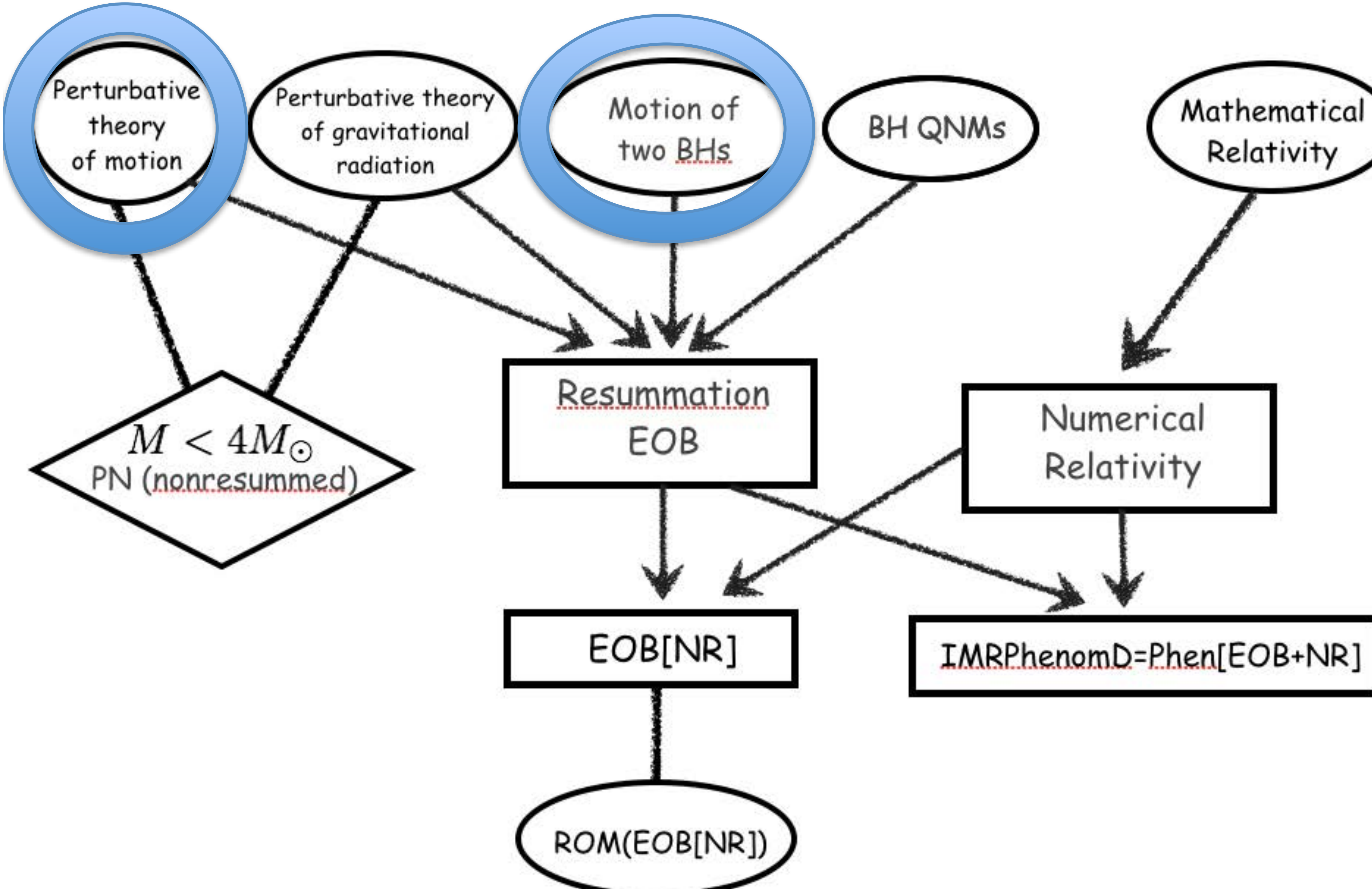


Damour-Deruelle'81 Damour'82

EOM (v/c)^5

$$\frac{dP}{dt} = -\frac{192\pi}{5c^5}\left(\frac{2\pi G}{P}\right)^{\frac{5}{3}}\mu M^{\frac{2}{3}}\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{\frac{7}{2}}}$$


Perturbative theory of motion
Perturbative theory of gravitational radiation
Motion of two BHs
BH QNMs
Mathematical Relativity
$M < 4M_\odot$
PN (nonresummed)
Resummation EOB
Numerical Relativity
EOB[NR]
IMRPhenomD=Phen[EOB+NR]
ROM(EOB[NR])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Long History of the GR Problem of Motion

Einstein 1912 : geodesic principle $-\int m\sqrt{-g_{\mu\nu}\, dx^\mu\, dx^\nu}$

Einstein 1913-1916 post-Minkowskian $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\ ,\ h_{\mu\nu} \ll 1$

Einstein, Droste : post-Newtonian $h_{00} \sim h_{ij} \sim \frac{v^2}{c^2}\ ,\ h_{0i} \sim \frac{v^3}{c^3}\ ,\ \partial_0 h \sim \frac{v}{c}\partial_i h$

Weakly self-gravitating bodies:

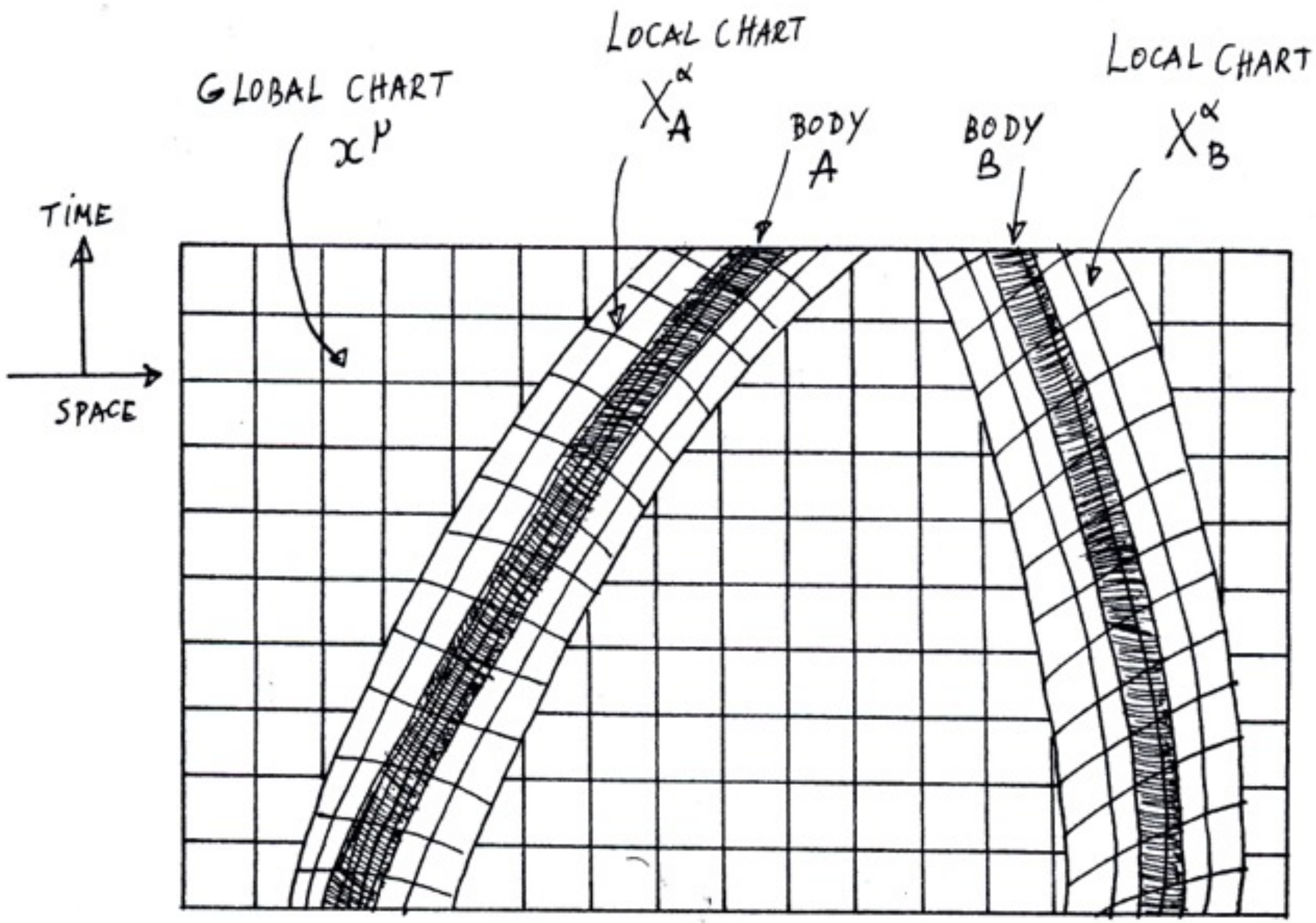
$$\nabla_\nu\, T^{\mu\nu} = 0 \quad ; \quad T^{\mu\nu} = \rho'\, u^\mu u^\nu + p\, g^{\mu\nu} \Rightarrow \nabla_u\, u^\mu = O(\nabla p)$$

Einstein-Grossmann '13,
1916 post-Newtonian: Droste, Lorentz, Einstein (visiting Leiden), De Sitter ;
Lorentz-Droste '17, Chazy '28, Levi-Civita '37 ….,

Eddington' 21, …, Lichnerowicz '39, Fock '39, Papapetrou '51,
… Dixon '64, Bailey-Israël '75, Ehlers-Rudolph '77….

- Multi-chart approach and matched asymptotic expansions: necessary for strongly self-gravitating bodies (NS, BH) Manasse (Wheeler) ‘63, Demianski-Grishchuk ‘74, D’Eath ‘75, Kates ‘80, Damour ‘82

Useful even for weakly self-gravitating bodies, i.e.“relativistic celestial mechanics”, Brumberg-Kopeikin ’89, Damour-Soffel-Xu ‘91-94



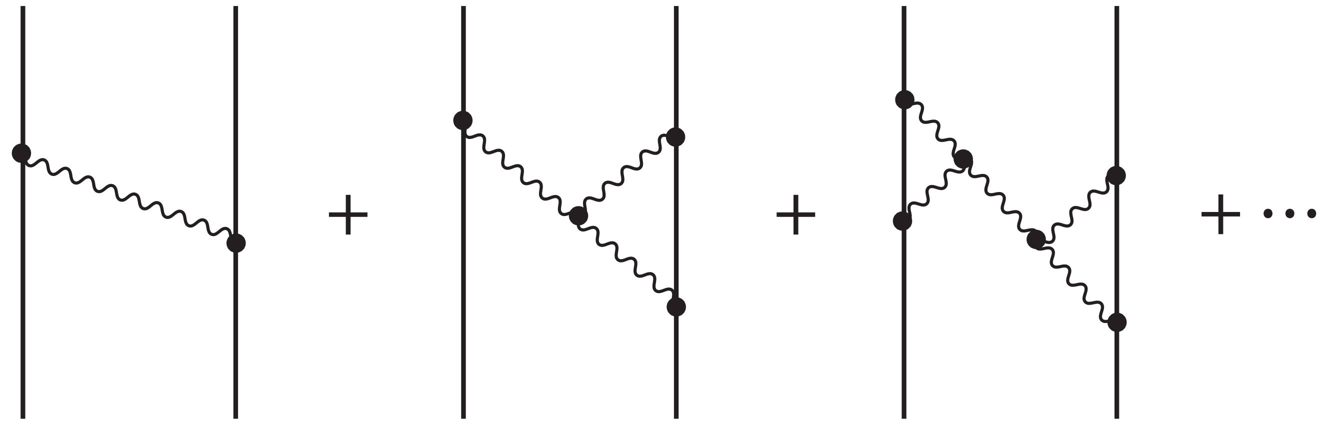
Practical Techniques for Computing the Motion of Compact Bodies (NS or BH)

Skeletonization : $T_{\mu\nu} \longrightarrow$ point-masses (Mathisson '31)

delta-functions in GR : Infeld '54, Infeld-Plebanski '60

justified by Matched Asymptotic Expansions (« Effacing Principle » Damour '83)

QFT's analytic (Riesz '49) or dimensional regularization (Bollini-Giambiagi '72, t'Hooft-Veltman '72) imported in GR (Damour '80, Damour-Jaranowski-Schäfer '01, …)



Feynman-like diagrams and

« Effective Field Theory » techniques

Bertotti-Plebanski '60,

Damour-Esposito-Farèse '96,

Goldberger-Rothstein '06, Porto '06, Gilmore-Ross' 08, Levi '10,

Foffa-Sturani '11 '13, Levi-Steinhoff '14, '15

Reduced (Fokker 1929) Action for Conservative Dynamics

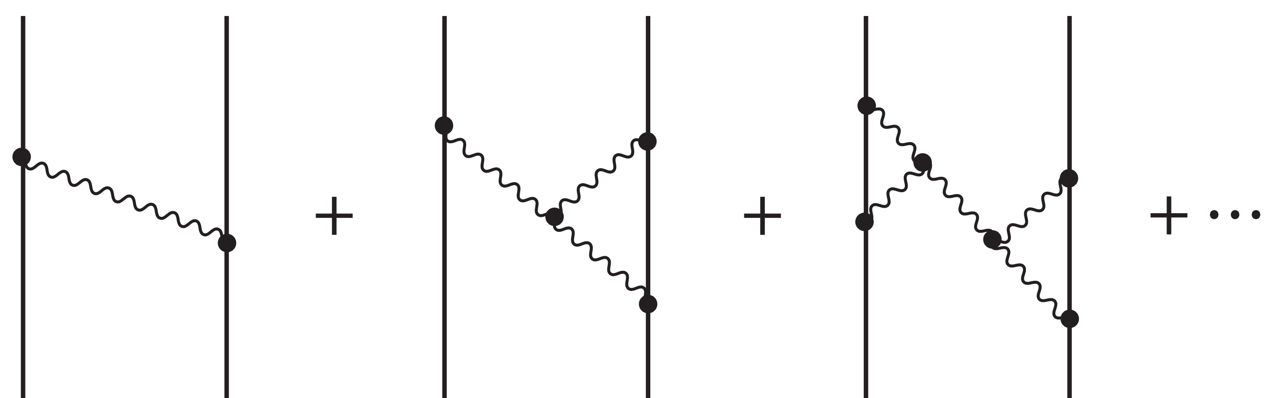
Needs gauge-fixed* action and time-symmetric Green function G.
*E.g. Arnowitt-Deser-Misner Hamiltonian formalism or harmonic coordinates.
Perturbatively solving (in dimension D=4 - eps) Einstein’s equations
to get the equations of motion and the action for the conservative dynamics

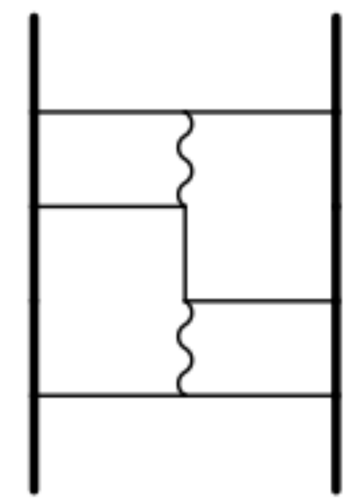
$$g = \eta + h$$

$$S(h,T) = \int \left(\frac{1}{2} h \Box h + \partial\partial hhh + ... + (h + hh + ...)T \right)$$

$$\Box h = -T + ... \rightarrow h = GT + ...$$

$$S_{\rm red}(T) = \frac{1}{2} TGT + V_3(GT, GT, GT) + ...$$





Beyond 1-loop order needs to use PN-expanded Green function for explicit computations.
Introduces IR divergences on top of the UV divergences linked to the point-particle description.
UV is (essentially) finite in dim.reg. and IR is linked to 4PN non-locality (Blanchet-Damour ’88).

$$\Box^{-1} = (\Delta - \frac{1}{c^2}\partial_t^2)^{-1} = \Delta^{-1} + \frac{1}{c^2}\partial_t^2 \Delta^{-2} + ...$$

Post-Newtonian Equations of Motion [2-body, wo spins]

- 1PN (including v^2/c^2) [Lorentz-Droste '17], Einstein-Infeld-Hoffmann '38
- 2PN (inc. v^4/c^4) Ohta-Okamura-Kimura-Hiida '74, Damour-Deruelle '81
  Damour '82, Schäfer '85, Kopeikin '85
- 2.5 PN (inc. v^5/c^5) Damour-Deruelle '81, Damour '82, Schäfer '85,
  Kopeikin '85
- 3 PN (inc. v^6/c^6) Jaranowski-Schäfer '98, Blanchet-Faye '00,
  Damour-Jaranowski-Schäfer '01, Itoh-Futamase '03,
  Blanchet-Damour-Esposito-Farèse' 04, Foffa-Sturani '11
- 3.5 PN (inc. v^7/c^7) Iyer-Will '93, Jaranowski-Schäfer '97, Pati-Will '02,
  Königsdörffer-Faye-Schäfer '03, Nissanke-Blanchet '05, Itoh '09
- 4PN (inc. v^8/c^8) Jaranowski-Schäfer '13, Foffa-Sturani '13,'16
  Bini-Damour '13, Damour-Jaranowski-Schäfer '14, Bernard et al'16

New feature : non-locality in time

2-body Taylor-expanded N + 1PN + 2PN Hamiltonian

$$H_{\mathrm{N}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{1}{2}\frac{Gm_1m_2}{r_{12}} + (1 \leftrightarrow 2)$$

$$c^2 H_{1\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(-12\frac{\mathbf{p}_1^2}{m_1^2} + 14\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + 2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$+\frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}} + (1 \leftrightarrow 2),$$

$$c^4 H_{2\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(5\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} - \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + 5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right.$$
$$\left. - 6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} - \frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right)$$
$$+\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2} + 19\frac{\mathbf{p}_2^2}{m_2^2}\right) - \frac{1}{2}(m_1+m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2) + 6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$-\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2 + 5m_1m_2 + m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2),$$

2-body Taylor-expanded 3PN Hamiltonian [JS 98, DJS 01]

$$
\begin{aligned}
c^6 H_{3\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a) = & -\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32}\frac{G m_1 m_2}{r_{12}}\Bigg(-14\frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 4\mathbf{p}_1^2\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4 m_2^2} + 6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4 m_2^2} \\
& - 10\frac{(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2 + \mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4 m_2^2} + 24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4 m_2^2} \\
& + 2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3 m_2^3} + \frac{(7\mathbf{p}_1^2\mathbf{p}_2^2 - 10(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2^3} \\
& + \frac{(\mathbf{p}_1^2\mathbf{p}_2^2 - 2(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2^3} + 15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3 m_2^3} \\
& - 18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3 m_2^3} + 5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3 m_2^3}\Bigg) + \frac{G^2 m_1 m_2}{r_{12}^2}\Bigg(\frac{1}{16}(m_1 - 27 m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4} \\
& - \frac{115}{16} m_1 \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2} + \frac{1}{48} m_2 \frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 371\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2} + \frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3} + \frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3} \\
& - \frac{1}{8} m_1 \frac{(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2) + 11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3 m_2} - \frac{3}{2} m_1 \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2} \\
& + \frac{125}{12} m_2 \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2 m_2^2} + \frac{10}{3} m_2 \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} \\
& - \frac{1}{48}(220 m_1 + 193 m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\Bigg) + \frac{G^3 m_1 m_2}{r_{12}^3}\Bigg(-\frac{1}{48}\left(425 m_1^2 + \left(473 - \frac{3}{4}\pi^2\right) m_1 m_2 + 150 m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2} \\
& + \frac{1}{16}\left(77(m_1^2 + m_2^2) + \left(143 - \frac{1}{4}\pi^2\right) m_1 m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \frac{1}{16}\left(20 m_1^2 - \left(43 + \frac{3}{4}\pi^2\right) m_1 m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} \\
& + \frac{1}{16}\left(21(m_1^2 + m_2^2) + \left(119 + \frac{3}{4}\pi^2\right) m_1 m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\Bigg) \\
& + \frac{1}{8}\frac{G^4 m_1 m_2^3}{r_{12}^4}\left(\left(\frac{227}{3} - \frac{21}{4}\pi^2\right) m_1 + m_2\right) + (1 \leftrightarrow 2).
\end{aligned}
$$

2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014]

$$
\begin{aligned}
c^8 H^{\rm local}_{\rm 4PN}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{7(\mathbf{p}_1^2)^5}{256 m_1^9} + \frac{G m_1 m_2}{r_{12}} H_{48}(\mathbf{x}_a,\mathbf{p}_a) + \frac{G^2 m_1 m_2}{r_{12}^2} m_1 H_{46}(\mathbf{x}_a,\mathbf{p}_a) \\
& + \frac{G^3 m_1 m_2}{r_{12}^3}\left(m_1^2 H_{441}(\mathbf{x}_a,\mathbf{p}_a) + m_1 m_2 H_{442}(\mathbf{x}_a,\mathbf{p}_a)\right) \\
& + \frac{G^4 m_1 m_2}{r_{12}^4}\left(m_1^3 H_{421}(\mathbf{x}_a,\mathbf{p}_a) + m_1^2 m_2 H_{422}(\mathbf{x}_a,\mathbf{p}_a)\right) \\
& + \frac{G^5 m_1 m_2}{r_{12}^5} H_{40}(\mathbf{x}_a,\mathbf{p}_a) + (1 \leftrightarrow 2). \qquad \text{(A3)}
\end{aligned}
$$

$$
\begin{aligned}
H_{48}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{45(\mathbf{p}_1^2)^4}{128 m_1^8} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64 m_1^6 m_2^2} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16 m_1^6 m_2^2} \\
& - \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{35(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{256 m_1^5 m_2^3} \\
& + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{128 m_1^5 m_2^3} + \frac{33(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256 m_1^5 m_2^3} - \frac{85(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3} \\
& - \frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2^3} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3} + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3} \\
& + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3} + \frac{55(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} \\
& - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128 m_1^5 m_2^3} - \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256 m_1^5 m_2^3} - \frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} \\
& + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{128 m_1^5 m_2^3} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{64 m_1^4 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64 m_1^4 m_2^4} \\
& - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{4 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^4 m_2^4} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64 m_1^4 m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64 m_1^4 m_2^4} \\
& - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{4 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{16 m_1^4 m_2^4} \\
& - \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64 m_1^4 m_2^4} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32 m_1^4 m_2^4} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128 m_1^4 m_2^4}. \qquad \text{(A4a)}
\end{aligned}
$$

$$
\begin{aligned}
H_{46}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{369(\mathbf{n}_{12}\cdot\mathbf{p}_1)^6}{160 m_1^6} - \frac{889(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_1^2}{192 m_1^6} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16 m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64 m_1^6} - \frac{549(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{128 m_1^5 m_2} \\
& + \frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{16 m_1^5 m_2} - \frac{167(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2}{128 m_1^5 m_2} + \frac{1547(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2} - \frac{851(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2} \\
& + \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2} + \frac{3263(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2} + \frac{1067(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{480 m_1^4 m_2^2} - \frac{4567(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840 m_1^4 m_2^2} \\
& - \frac{3571(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{320 m_1^4 m_2^2} + \frac{3073(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{480 m_1^4 m_2^2} + \frac{4349(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2} \\
& - \frac{3461\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{3840 m_1^4 m_2^2} + \frac{1673(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{1920 m_1^4 m_2^2} - \frac{1999(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2} - \frac{13(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8 m_1^3 m_2^3} \\
& + \frac{191(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{192 m_1^3 m_2^3} - \frac{19(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3} \\
& + \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{192 m_1^3 m_2^3} + \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{96 m_1^3 m_2^3} + \frac{233(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{96 m_1^3 m_2^3} - \frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32 m_1^3 m_2^3} \\
& + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{4 m_1^2 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{4 m_1^2 m_2^4} \\
& - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{2 m_1^2 m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^2 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{6 m_1^2 m_2^4} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48 m_1^2 m_2^4} \\
& - \frac{133(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{24 m_1^2 m_2^4} - \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{96 m_1^2 m_2^4} + \frac{197(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96 m_1^2 m_2^4} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48 m_1^2 m_2^4} + \frac{13(\mathbf{p}_2^2)^3}{8 m_2^6}. \qquad \text{(A4b)}
\end{aligned}
$$

$$
\begin{aligned}
H_{441}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{5027(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{384 m_1^4} - \frac{22993(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{960 m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152 m_1^4} - \frac{3191(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{640 m_1^3 m_2} \\
& + \frac{28561(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920 m_1^3 m_2} + \frac{8777(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2} + \frac{752969\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{28800 m_1^3 m_2} \\
& - \frac{16481(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{960 m_1^2 m_2^2} + \frac{94433(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{4800 m_1^2 m_2^2} - \frac{103957(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{2400 m_1^2 m_2^2} \\
& + \frac{791(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{400 m_1^2 m_2^2} + \frac{26627(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{1600 m_1^2 m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800 m_1^2 m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32 m_2^4}, \qquad \text{(A4c)}
\end{aligned}
$$

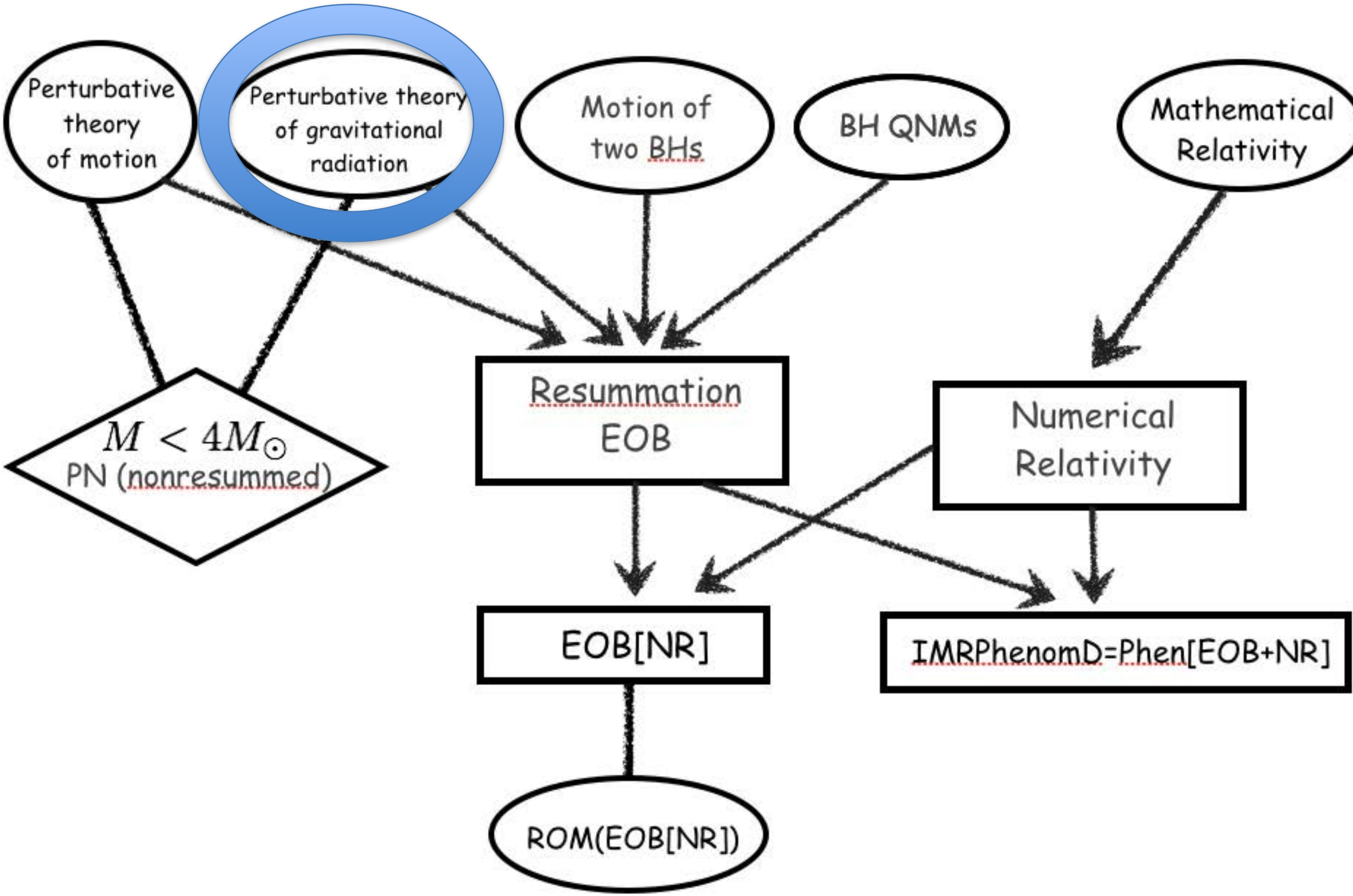
$$
\begin{aligned}
H_{442}(\mathbf{x}_a,\mathbf{p}_a) = {} & \left(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\right)\frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^4} \\
& + \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\right)\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2} \\
& + \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2 m_2^2} + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^2 m_2^2} \\
& - \left(\frac{30383}{960} + \frac{36405\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\right)\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2} \\
& + \left(\frac{2369}{60} + \frac{35655\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2} + \left(\frac{43101\pi^2}{16384} - \frac{391711}{6400}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{m_1^3 m_2} \\
& + \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2}. \qquad \text{(A4d)}
\end{aligned}
$$

$$
H_{421}(\mathbf{x}_a,\mathbf{p}_a) = \frac{64861\mathbf{p}_1^2}{4800 m_1^2} - \frac{91(\mathbf{p}_1\cdot\mathbf{p}_2)}{8 m_1 m_2} + \frac{105\mathbf{p}_2^2}{32 m_2^2} - \frac{9841(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600 m_1^2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2 m_1 m_2}. \qquad \text{(A4e)}
$$

$$
\begin{aligned}
H_{422}(\mathbf{x}_a,\mathbf{p}_a) = {} & \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\right)\frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\right)\frac{\mathbf{p}_2^2}{m_2^2} \\
& + \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} + \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2} \\
& + \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2}. \qquad \text{(A4f)}
\end{aligned}
$$

$$
H_{40}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\right)m_1^3 m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\right)m_1^2 m_2^2. \qquad \text{(A4g)}
$$

$$
\begin{aligned}
H^{\rm nonloc}_{\rm 4PN}(t) = {} & -\frac{1}{5}\frac{G^2 M}{c^8} I^{(3)}_{ij}(t) \\
& \times \mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{dv}{|v|} I^{(3)}_{ij}(t+v),
\end{aligned}
$$



Perturbative theory of motion
Perturbative theory of gravitational radiation
Motion of two BHs
BH QNMs
Mathematical Relativity
$M < 4M_\odot$
PN (nonresummed)
Resummation
EOB
Numerical Relativity
EOB[NR]
IMRPhenomD=Phen[EOB+NR]
ROM(EOB[NR])

Perturbative Theory of the Generation of Gravitational Radiation

Einstein ’16, ’18 (+ Landau-Lifshitz 41, and Fock ’55) : h_+, h_x and quadrupole formula

Relativistic, multipolar extensions of LO quadrupole radiation :

Sachs-Bergmann ’58, Sachs ’61, Mathews ’62, Peters-Mathews ’63, Pirani '64

Bonnor-Rotenberg ’66,

Campbell-Morgan ’71,

Campbell et al ’75,

Epstein-Wagoner-Will ’75-76

Thorne ’80, .., Will et al 00

MPM Formalism:

Blanchet-Damour ’86,

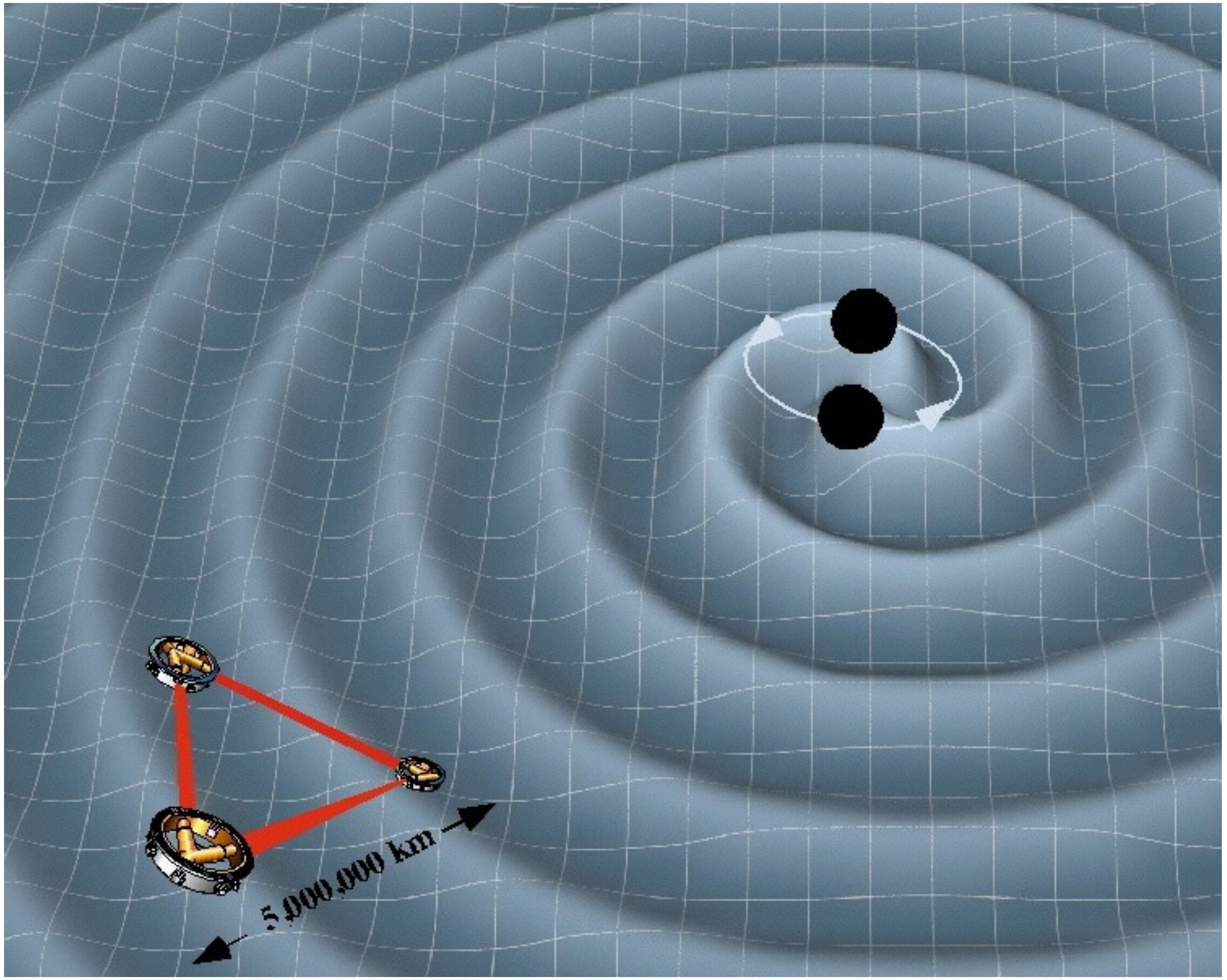
Damour-Iyer ’91,

Blanchet ’95 ‘98

Combines multipole exp.

Post Minkowkian exp.

and analytic continuation



MULTIPOLAR POST-MINKOWSKIAN FORMALISM (BLANCHET-DAMOUR-IYER)

Decomposition of space-time in various overlapping regions:

1. near-zone: r << lambda : PN theory
2. exterior zone: r >> r_source: MPM expansion
3. far wave-zone: Bondi-type expansion

followed by matching between the zones

in exterior zone, iterative solution of Einstein's vacuum field equations by means of a double expansion in non-linearity and in multipoles, with crucial use of analytic continuation (complex B) for dealing with formal UV divergences at r=0

$$g = \eta + G h_1 + G^2 h_2 + G^3 h_3 + \ldots,$$
$$\Box h_1 = 0,$$
$$\Box h_2 = \partial\partial h_1 h_1,$$
$$\Box h_3 = \partial\partial h_1 h_1 h_1 + \partial\partial h_1 h_2,$$
$$h_1 = \sum_\ell \partial_{i_1 i_2 \ldots i_\ell} \left(\frac{M_{i_1 i_2 \ldots i_\ell}(t - r/c)}{r} \right) + \partial\partial \ldots \partial \left(\frac{\epsilon_{j_1 j_2 k} S_{k j_3 \ldots j_\ell}(t - r/c)}{r} \right),$$
$$h_2 = FP_B \Box^{-1}_{\rm ret} \left(\left(\frac{r}{r_0} \right)^B \partial\partial h_1 h_1 \right) + \ldots,$$
$$h_3 = FP_B \Box^{-1}_{\rm ret} \ldots.$$

Perturbative computation of GW flux from binary system

- lowest order : Einstein 1918 Peters-Mathews 63
- 1 + (v^2/c^2) : Wagoner-Will 76
- … + (v^3/c^3) : Blanchet-Damour 92, Wiseman 93
- … + (v^4/c^4) : Blanchet-Damour-Iyer Will-Wiseman 95
- … + (v^5/c^5) : Blanchet 96
- … + (v^6/c^6) : Blanchet-Damour-Esposito-Farèse-Iyer 2004
- … + (v^7/c^7) : Blanchet

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

$$x = \left(\frac{v}{c}\right)^2 = \left(\frac{G(m_1+m_2)\Omega}{c^3}\right)^{\frac{2}{3}} = \left(\frac{\pi G(m_1+m_2)f}{c^3}\right)^{\frac{2}{3}}$$

$$\begin{aligned}\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} \\ & + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\ & + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E - \frac{856}{105}\ln(16\,x)\right. \\ & \quad \left. + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right] x^3 \\ & + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}.\end{aligned}$$

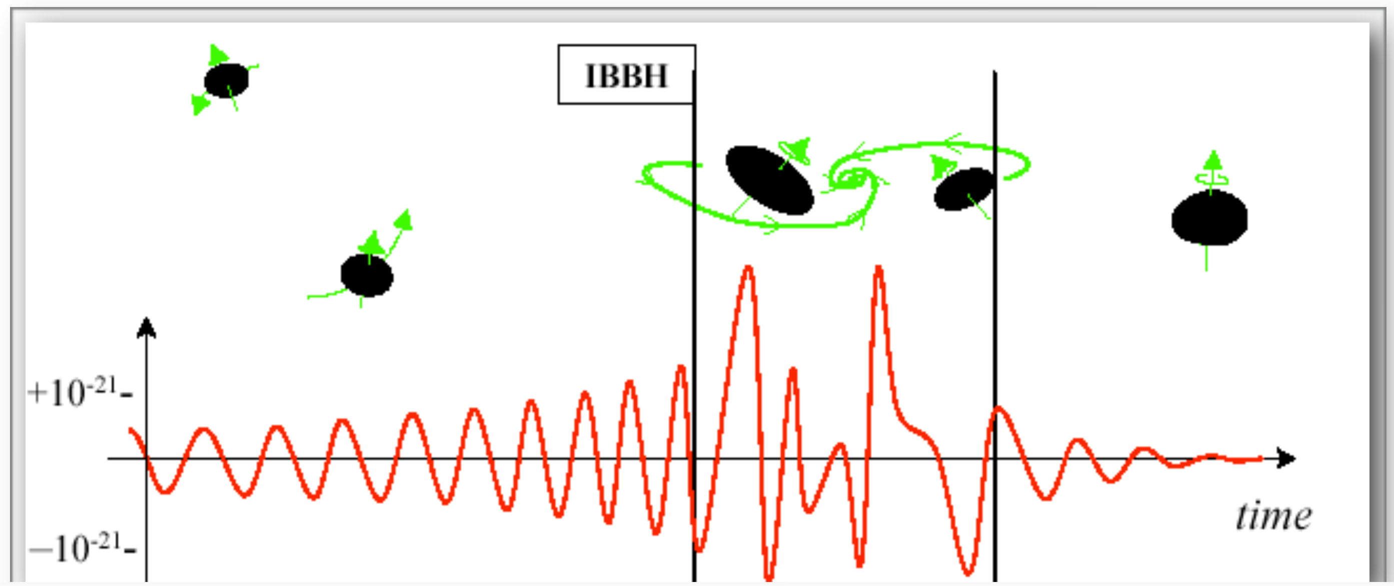
Analytical GW Templates for BBH Coalescences ?

PN corrections to Einstein's quadrupole frequency « chirping »
from PN-improved balance equation dE(f)/dt = - F(f)

$$\frac{d\phi}{d\ln f} = \frac{\omega^2}{d\omega/dt} = Q_\omega^N \widehat{Q}_\omega \qquad \frac{v}{c} = \left(\frac{\pi G(m_1+m_2)f}{c^3}\right)^{\frac{1}{3}}$$

$$Q_\omega^N = \frac{5\,c^5}{48\,\nu\,v^5}\,;\ \widehat{Q}_\omega = 1 + c_2\left(\frac{v}{c}\right)^2 + c_3\left(\frac{v}{c}\right)^3 + \cdots \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

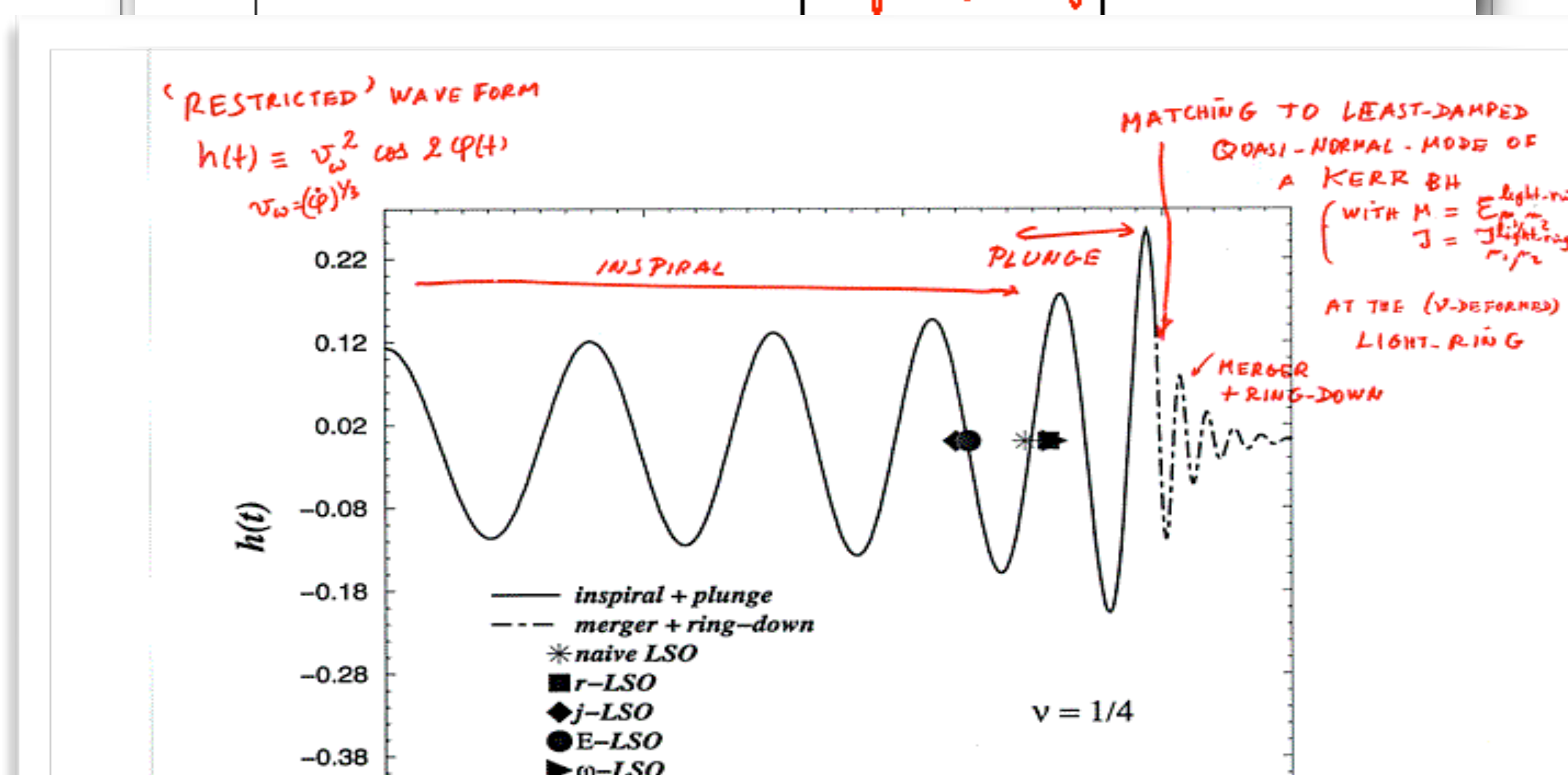
Cutler et al. '93:
« slow convergence of PN »

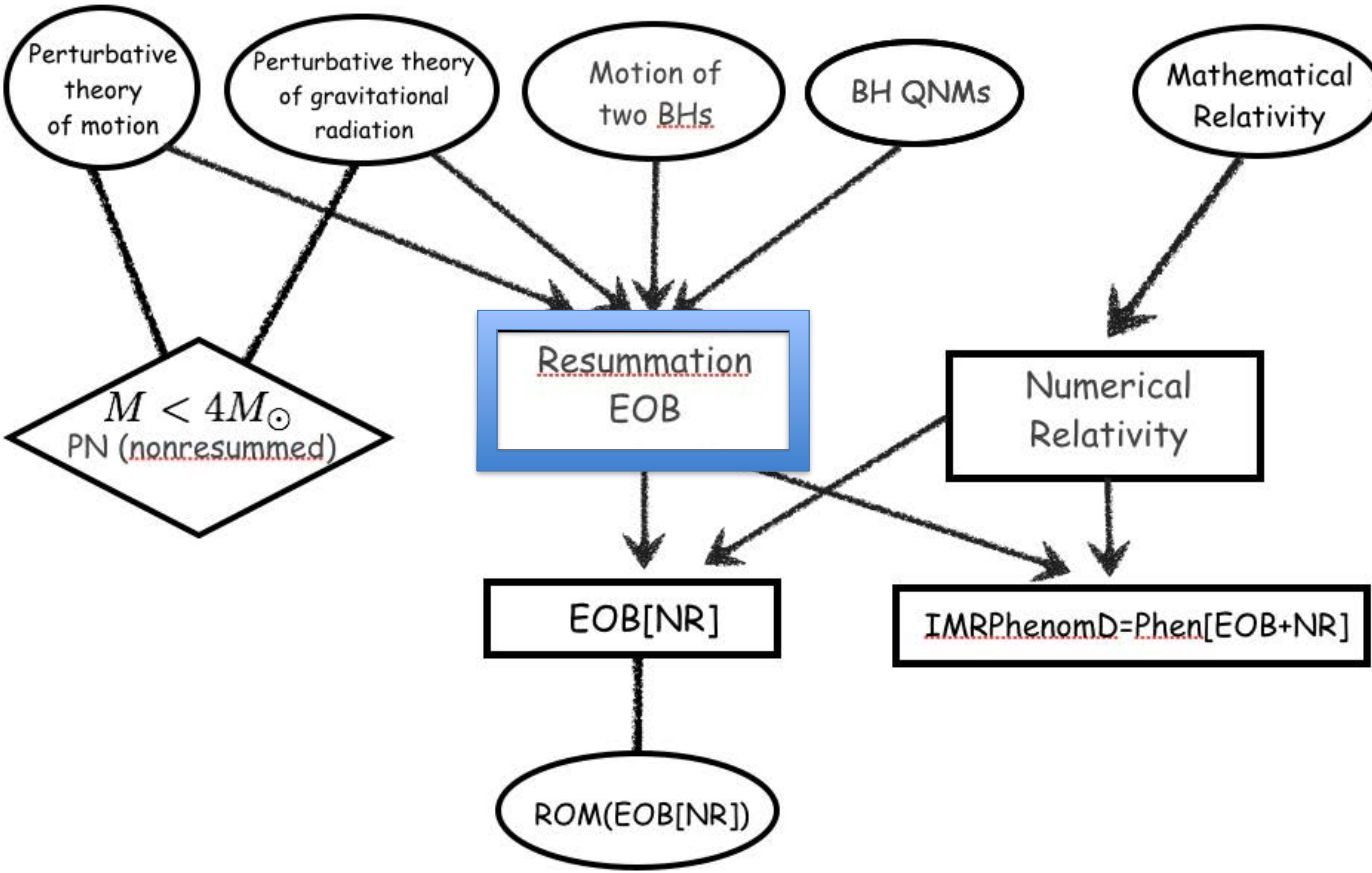


Brady-Creighton-Thorne'98:
« inability of current computational techniques to evolve a BBH through its last ~10 orbits of inspiral » and to compute the merger

Damour-Iyer-Sathyaprakash'98:
use resummation methods for E and F

Buonanno-Damour '99-00:
novel, resummed approach:
Effective-One-Body analytical formalism

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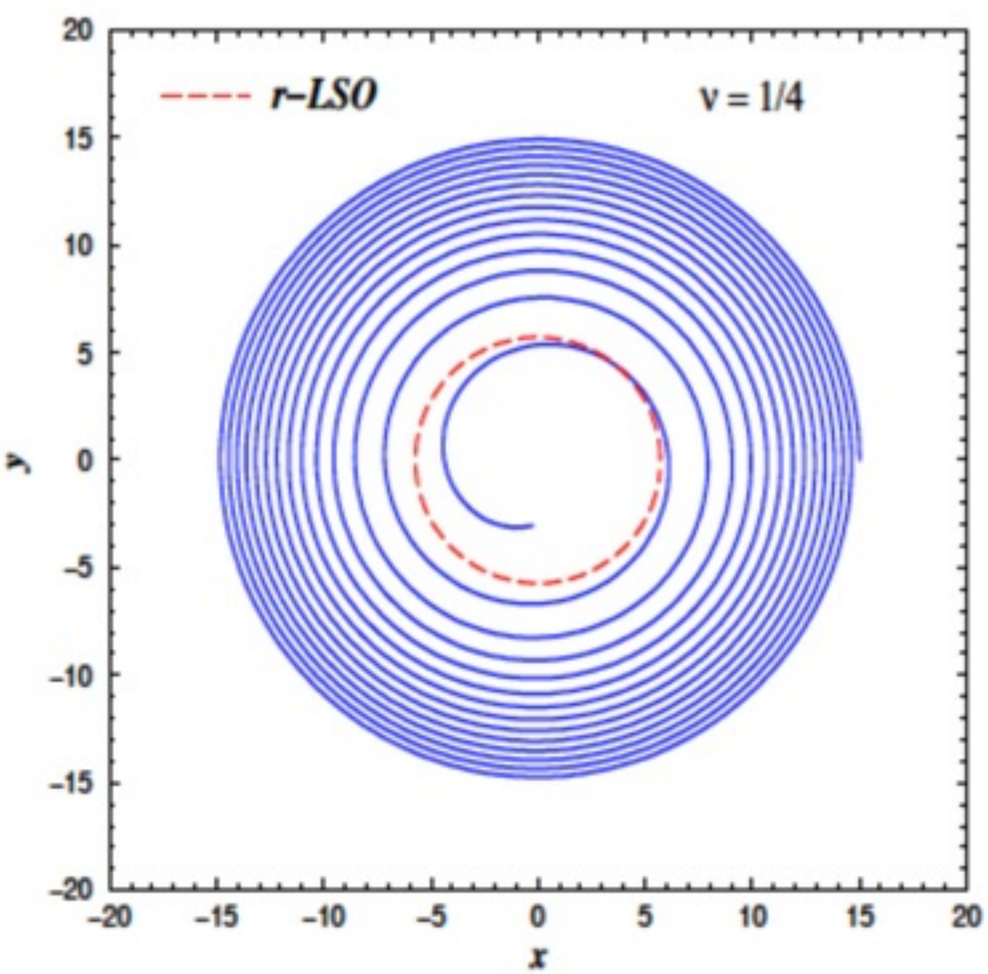
EOB[NR]
IMRPhenomD=Phen[EOB+NR]
ROM(EOB[NR])

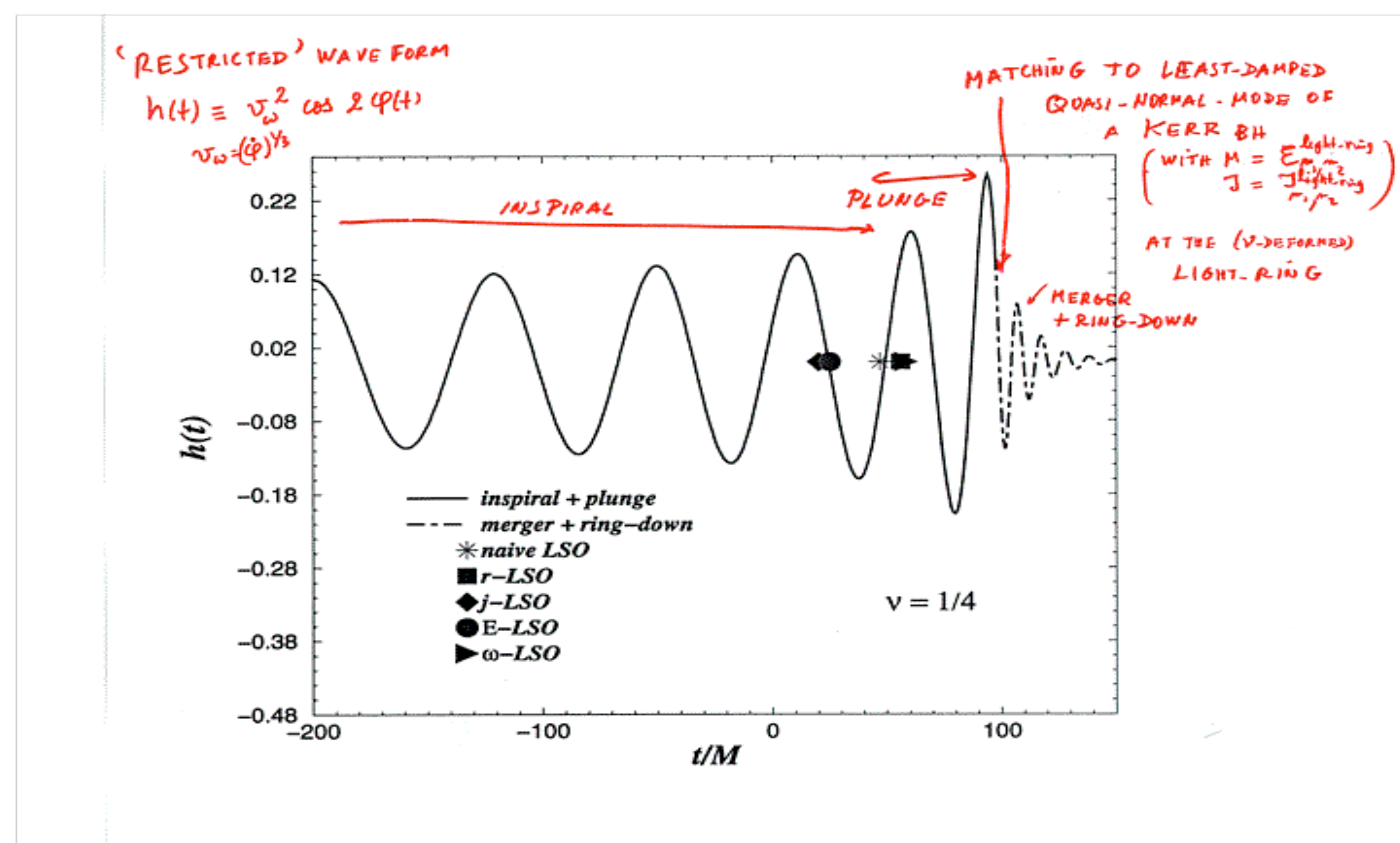
Effective One Body (EOB) Method)

Buonanno-Damour 1999, 2000; Damour-Jaranowski-Schaefer 2000; Damour 2001

Resummation of perturbative PN results → description of the coalescence
+ addition of ringdown (Vishveshwara 70, Davis-Ruffini-Tiomno 1972)

Buonanno-Damour 2000

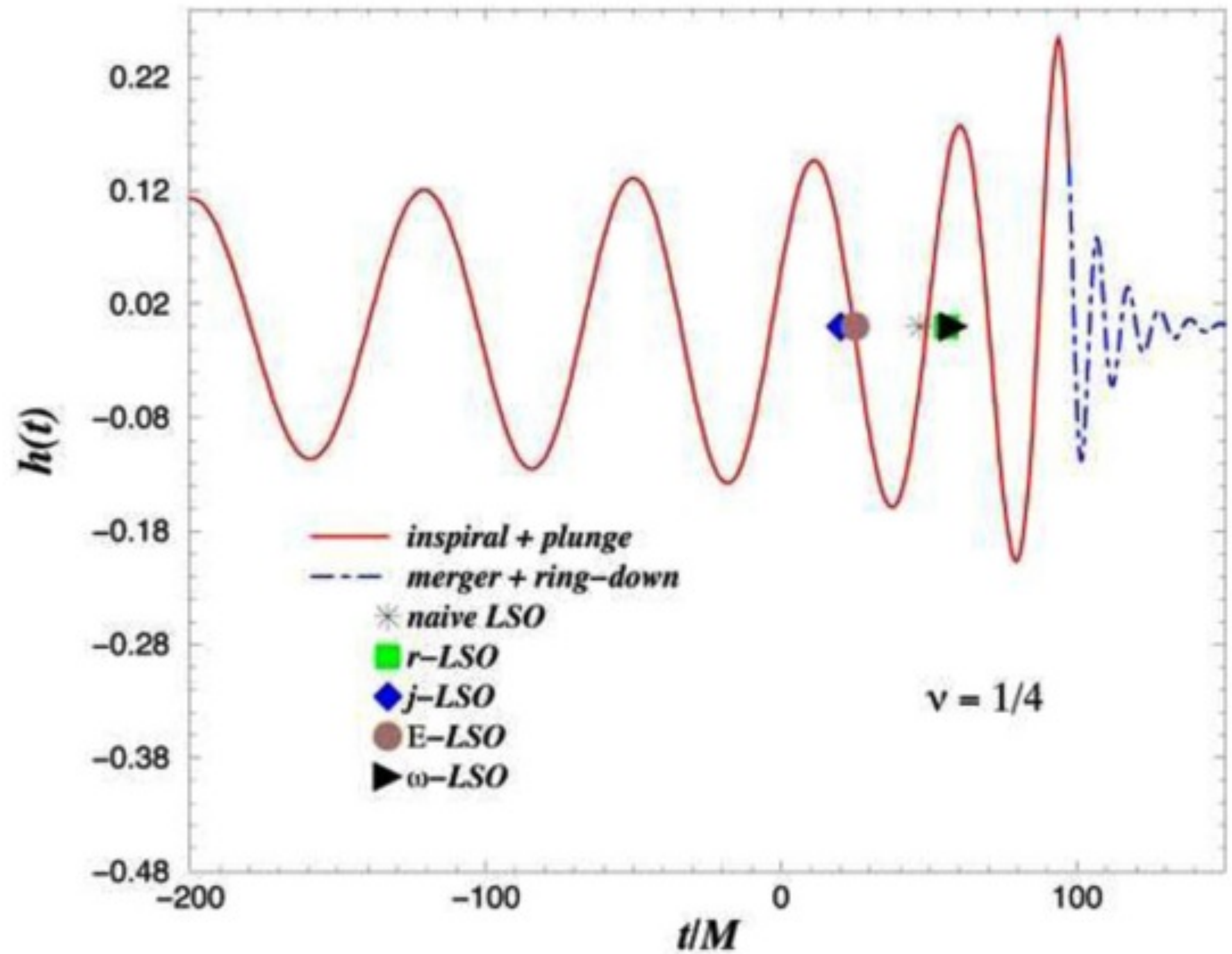




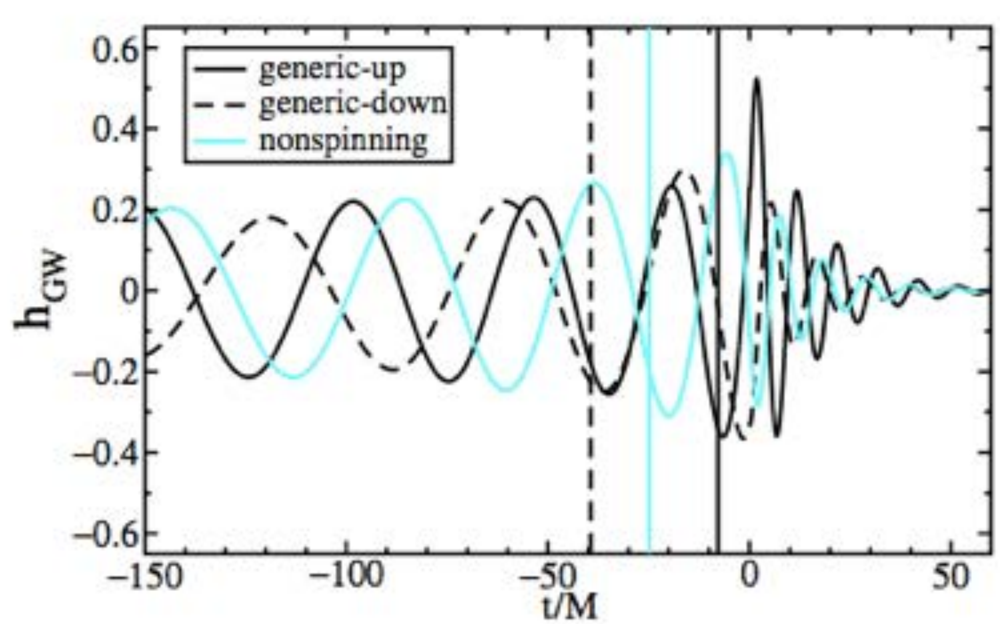
Predictions as early as 2000 :
continued transition, non adiabaticity, first complete waveform, final spin (OK within 10%), final mass

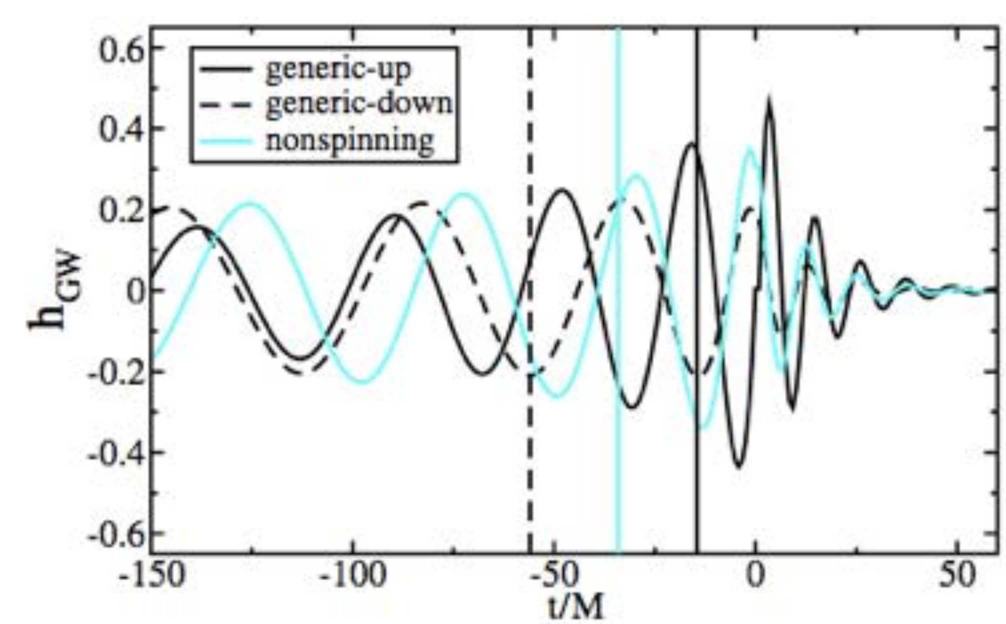
First complete waveforms for BBH coalescences: analytical EOB

Non-spinning BHs
Buonanno-Damour 2000



Spinning BHs
Buonanno-Chen-Damour
Nov 2005:
« to show the
promise
of a purely
analytical
EOB-based
approach »





EOB THEORY + EOB[NR] + EOB[SF] DEVELOPMENTS

Buonanno,Damour 99 (2 PN Hamiltonian)

Buonanno,Damour 00 (Rad.Reac. full waveform)

Damour, Jaranowski,Schäfer 00 (3 PN Hamiltonian)

Damour 01,
Buonanno, Chen, Damour 05,
Damour-Jaranowski,Schäfer 08,
Barausse, Buonanno, 10,
Nagar 11,
Balmelli-Jetzer 12,
Taracchini et al 12,14,
Damour,Nagar 14
(spinning bodies)

Damour, Nagar 07,
Damour, Iyer, Nagar 08,
Pan et al. 11
(factorized waveform)

Damour, Nagar 10
Bini-Damour-Faye 12 (tidal effects)

Bini, Damour 13, Damour, Jaranowski, Schäfer 15 (4 PN Hamiltonian)

EOB vs NR and EOB[NR]
Buonanno, Cook, Pretorius 07,
Buonanno, Pan, Taracchini 08-
Damour-Nagar 08-

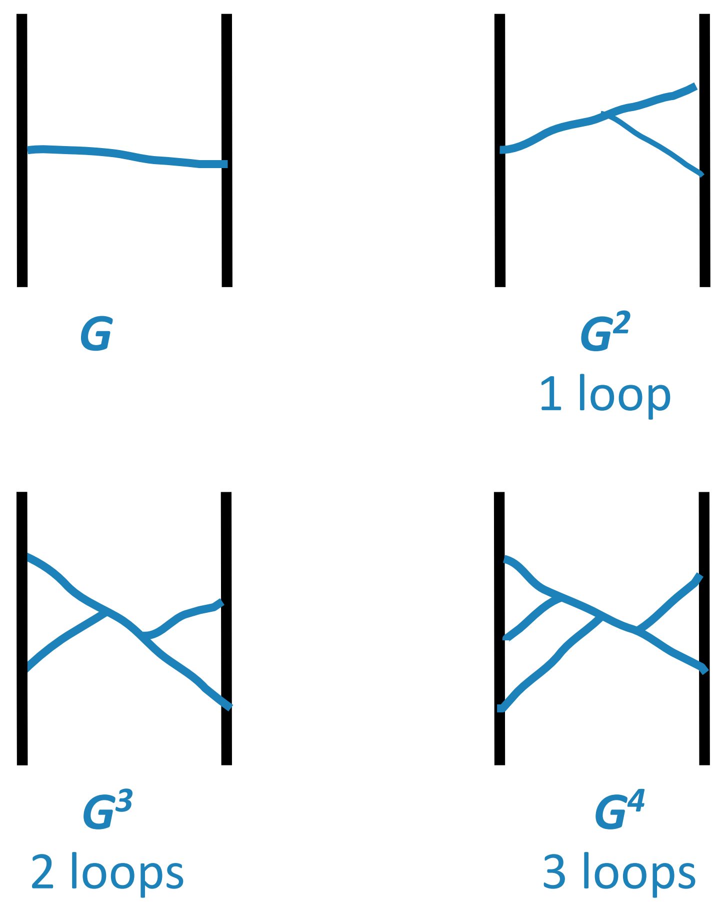
EOB vs SF and EOB[SF]
Damour 09
Barack-Sago-Damour 10
Barausse-Buonanno-LeTiec 12
Akcay-Barack-Damour-Sago 12
Bini-Damour 13-16
LeTiec 15
Bini-Damour-Geralico 16
Hopper-Kavanagh-Ottewill 16
Akcay-vandeMeent 16

Reduced Order Model version (Pürrer 2014, 2016) of EOB[NR] (Taracchini et al 2014)

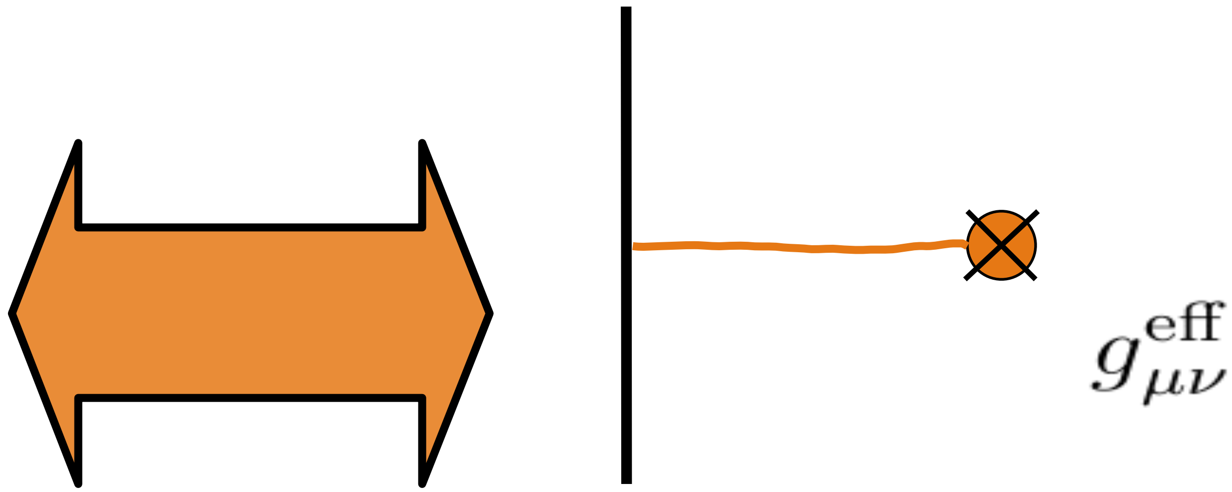
Phenomenological model (Ajith et al 2007, Hannam et al 2014, Husa et al 2016, Kahn et al 2016) of FFT of hybrids EOB + NR

Real dynamics versus Effective dynamics

Real dynamics



Effective dynamics



$$S = -\int \mu ds + \ldots$$

$$H = H_0 + \left(G H_1 + \frac{G^2}{c^2} H_2 + \frac{G^3}{c^4} H_3 + \frac{G^4}{c^6} H_4 \right) \left(1 + \frac{1}{c^2} + \ldots \right)$$

Effective metric for non-spinning bodies: a nu-deformation of Schwarzschild

$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

$$ds^2_{\mathrm{eff}} = -A(r;\nu)\, dt^2 + B(r;\nu)\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2 \right)$$

TWO-BODY/EOB "CORRESPONDENCE":

THINK QUANTUM-MECHANICALLY (J.A. WHEELER)

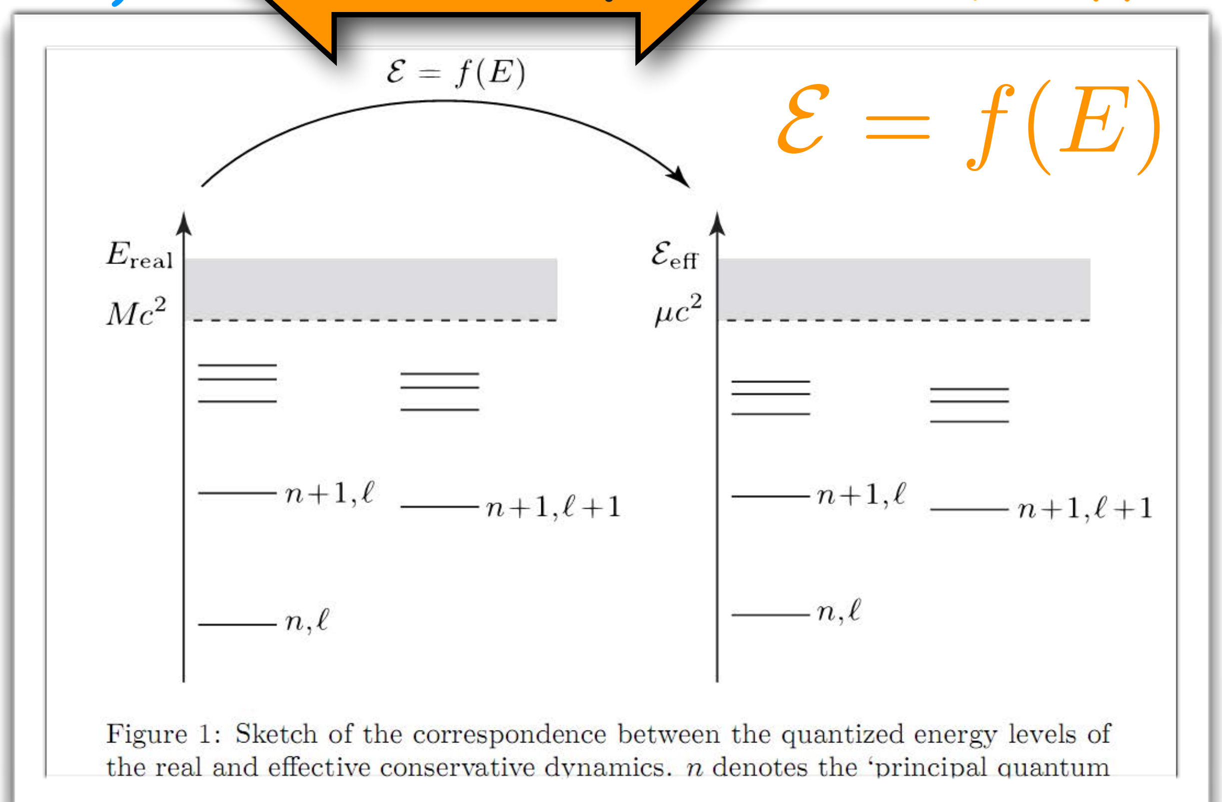
Real 2-body system (in the c.o.m. frame) (m_1, m_2)

1:1 map

An effective particle in some effective metric

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$g_{\mu\nu}^{\text{eff}}$$



$$\mathcal{E} = f(E)$$

$$\mu^2 + g_{\text{eff}}^{\mu\nu} \frac{\partial S_{\text{eff}}}{\partial x^\mu} \frac{\partial S_{\text{eff}}}{\partial x^\nu} + \mathcal{O}(p^4) = 0$$

Bohr-Sommerfeld's Quantization Conditions (action-angle variables & Delaunay Hamiltonian)

$$J = \ell\hbar = \frac{1}{2\pi} \oint p_\varphi d\varphi$$

$$N = n\hbar = I_r + J$$

$$I_r = \frac{1}{2\pi} \oint p_r dr$$

$$H^{\text{classical}}(q, p) \rightarrow H^{\text{classical}}(I_a) \rightarrow E^{\text{quantum}}(I_a = n_a h) = f^{-1}[\mathcal{E}_{\text{eff}}^{\text{quantum}}(I_a^{\text{eff}} = n_a h)]$$

2-body CoM Newton + 1PN + 2PN + 3PN Hamiltonian

$$\widehat{H}_{\rm N}(\mathbf{r},\mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{1}{r},$$

$$\widehat{H}_{\rm 1PN}(\mathbf{r},\mathbf{p}) = \frac{1}{8}(3\nu-1)(\mathbf{p}^2)^2 - \frac{1}{2}\left\{(3+\nu)\mathbf{p}^2 + \nu(\mathbf{n}\cdot\mathbf{p})^2\right\}\frac{1}{r} + \frac{1}{2r^2},$$

$$\begin{aligned}\widehat{H}_{\rm 2PN}(\mathbf{r},\mathbf{p}) &= \frac{1}{16}\left(1-5\nu+5\nu^2\right)(\mathbf{p}^2)^3 + \frac{1}{8}\left\{\left(5-20\nu-3\nu^2\right)(\mathbf{p}^2)^2 - 2\nu^2(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - 3\nu^2(\mathbf{n}\cdot\mathbf{p})^4\right\}\frac{1}{r}\\ &\quad + \frac{1}{2}\left\{(5+8\nu)\mathbf{p}^2 + 3\nu(\mathbf{n}\cdot\mathbf{p})^2\right\}\frac{1}{r^2} - \frac{1}{4}(1+3\nu)\frac{1}{r^3},\end{aligned}$$

$$\begin{aligned}\widehat{H}_{\rm 3PN}(\mathbf{r},\mathbf{p}) &= \frac{1}{128}\left(-5+35\nu-70\nu^2+35\nu^3\right)(\mathbf{p}^2)^4\\ &\quad + \frac{1}{16}\left\{\left(-7+42\nu-53\nu^2-5\nu^3\right)(\mathbf{p}^2)^3 + (2-3\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + 3(1-\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - 5\nu^3(\mathbf{n}\cdot\mathbf{p})^6\right\}\frac{1}{r}\\ &\quad + \left\{\frac{1}{16}\left(-27+136\nu+109\nu^2\right)(\mathbf{p}^2)^2 + \frac{1}{16}(17+30\nu)\nu(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \frac{1}{12}(5+43\nu)\nu(\mathbf{n}\cdot\mathbf{p})^4\right\}\frac{1}{r^2}\\ &\quad + \left\{\left(-\frac{25}{8} + \left(\frac{\pi^2}{64} - \frac{335}{48}\right)\nu - \frac{23\nu^2}{8}\right)\mathbf{p}^2 + \left(-\frac{85}{16} - \frac{3\pi^2}{64} - \frac{7\nu}{4}\right)\nu(\mathbf{n}\cdot\mathbf{p})^2\right\}\frac{1}{r^3}\\ &\quad + \left\{\frac{1}{8} + \left(\frac{109}{12} - \frac{21}{32}\pi^2\right)\nu\right\}\frac{1}{r^4}.\end{aligned}$$

Resummed (non-spinning) EOB Hamiltonian

$$ds^2_{\text{eff}} = -A(r;\nu)\,dt^2 + B(r;\nu)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)$$

$$H_{\text{EOB}} = M\sqrt{1 + 2\nu\left(\frac{1}{\mu}\sqrt{A(r)\left(\mu^2 + \frac{p_r^2}{B(r)} + \frac{p_\phi^2}{r^2} + 2\nu(4-3\nu)\left(\frac{GM}{r}\right)^2\frac{p_r^4}{\mu^2}\right)} - 1\right)}$$

$$A^{3PN}(r;M,\nu) = \text{Pade}^1_3\left[1 - 2\frac{GM}{c^2r} + 2\nu\left(\frac{GM}{c^2r}\right)^3 + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\nu\left(\frac{GM}{c^2r}\right)^4\right]$$

$$M = m_1 + m_2\,, \qquad \mu = \frac{m_1m_2}{m_1+m_2}\,, \qquad \nu = \frac{m_1m_2}{(m_1+m_2)^2} = \frac{\mu}{M}$$

Resummed EOB waveform

(Damour-Iyer-Sathyaprakash 1998) Damour-Nagar 2007, Damour-Iyer -Nagar 2008

$$h_{\ell m} \equiv h_{\ell m}^{(N,\epsilon)} \hat{h}_{\ell m}^{(\epsilon)} \hat{h}_{\ell m}^{\rm NQC}$$

$$\hat{h}_{\ell m}^{(\epsilon)} = \hat{S}_{\rm eff}^{(\epsilon)} T_{\ell m} e^{i\delta_{\ell m}} \rho_{\ell m}^{\ell}$$

$$T_{\ell m} = \frac{\Gamma(\ell + 1 - 2i\hat{\hat{k}})}{\Gamma(\ell+1)} e^{\pi \hat{k}} e^{2i\hat{\hat{k}} \ln(2kr_0)}$$

$$\rho_{22}(x;\nu) = 1 + \left(\frac{55\nu}{84} - \frac{43}{42}\right) x + \left(\frac{19583\nu^2}{42336} - \frac{33025\nu}{21168} - \frac{20555}{10584}\right) x^2$$
$$+ \left(\frac{10620745\nu^3}{39118464} - \frac{6292061\nu^2}{3259872} + \frac{41\pi^2\nu}{192} - \frac{48993925\nu}{9779616} - \frac{428}{105}\mathrm{eulerlog}_2(x) + \frac{1556919113}{122245200}\right) x^3$$
$$+ \left(\frac{9202}{2205}\mathrm{eulerlog}_2(x) - \frac{387216563023}{160190110080}\right) x^4 + \left(\frac{439877}{55566}\mathrm{eulerlog}_2(x) - \frac{16094530514677}{533967033600}\right) x^5 + \mathcal{O}(x^6),$$

$$\mathcal{F}_\varphi \equiv -\frac{1}{8\pi\Omega} \sum_{\ell=2}^{\ell_{\max}} \sum_{m=1}^{\ell} (m\Omega)^2 |R h_{\ell m}^{(\epsilon)}|^2$$

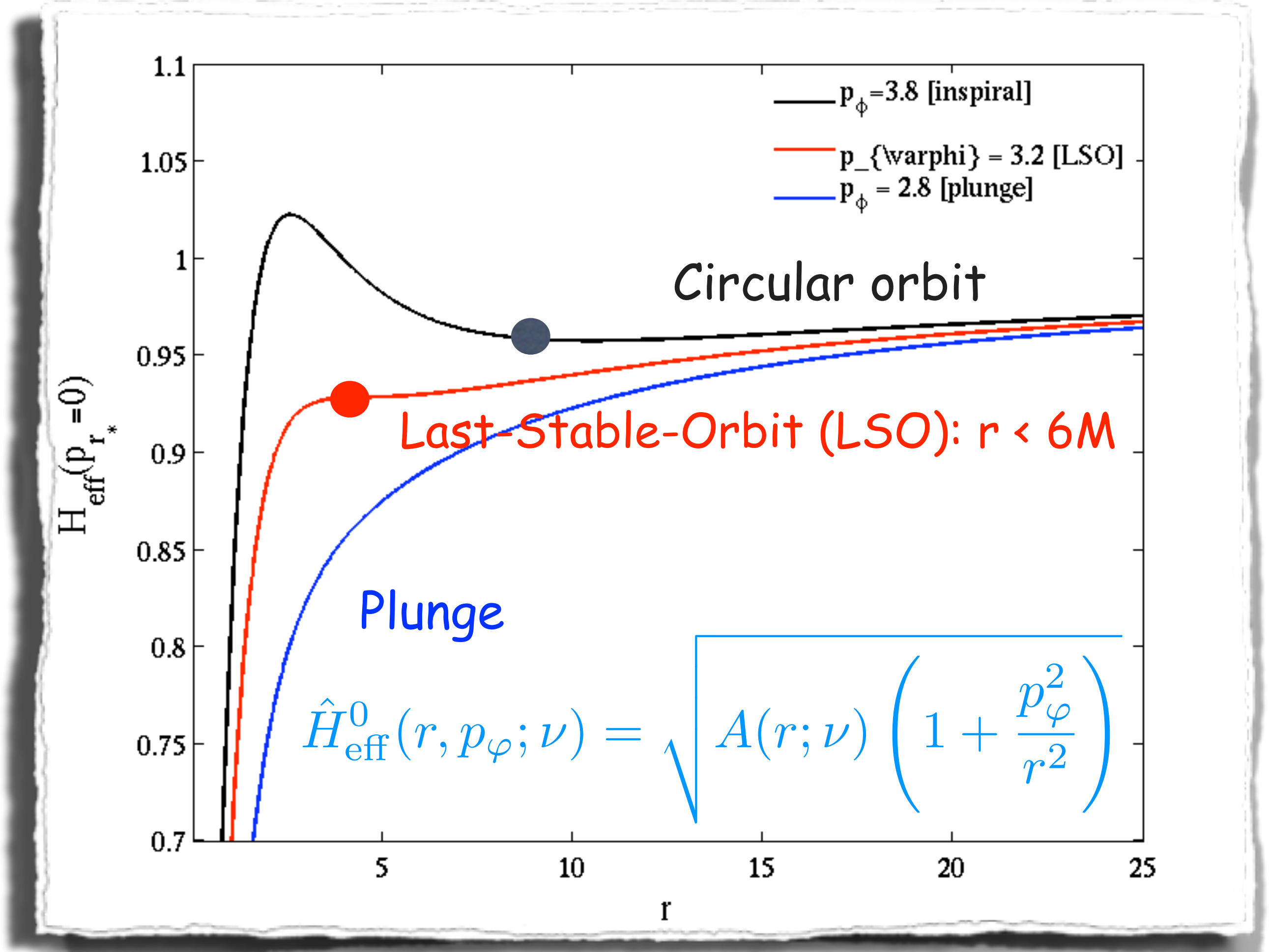
HAMILTON'S EQUATIONS & RADIATION REACTION

$$\dot{r} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial p_{r_*}}$$

$$\dot{\varphi} = \frac{\partial \hat{H}_{\rm EOB}}{\partial p_{\varphi}} \equiv \Omega$$

$$\dot{p}_{r_*} = -\left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial r} + \hat{\mathcal{F}}_{r_*}$$

$$\dot{p}_{\varphi} = \hat{\mathcal{F}}_{\varphi}$$

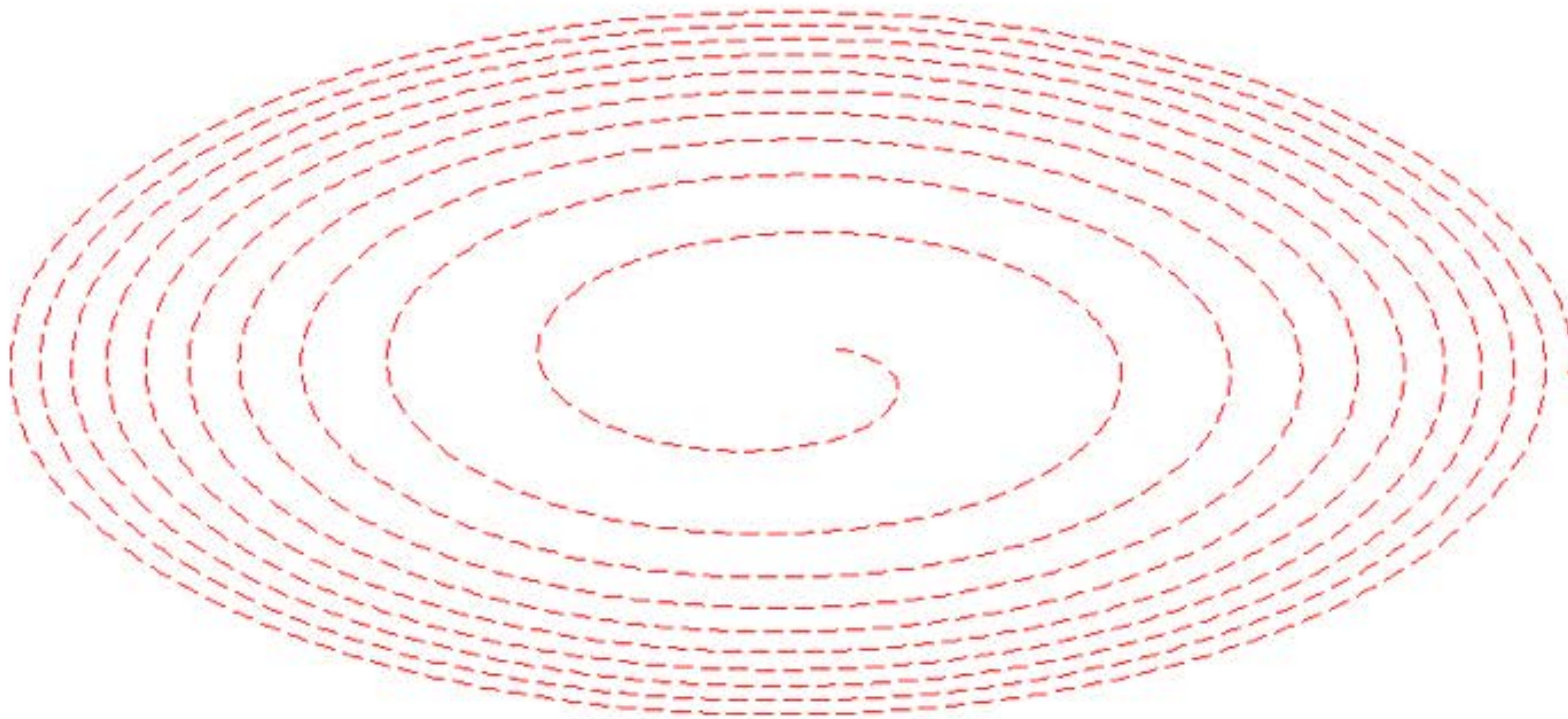


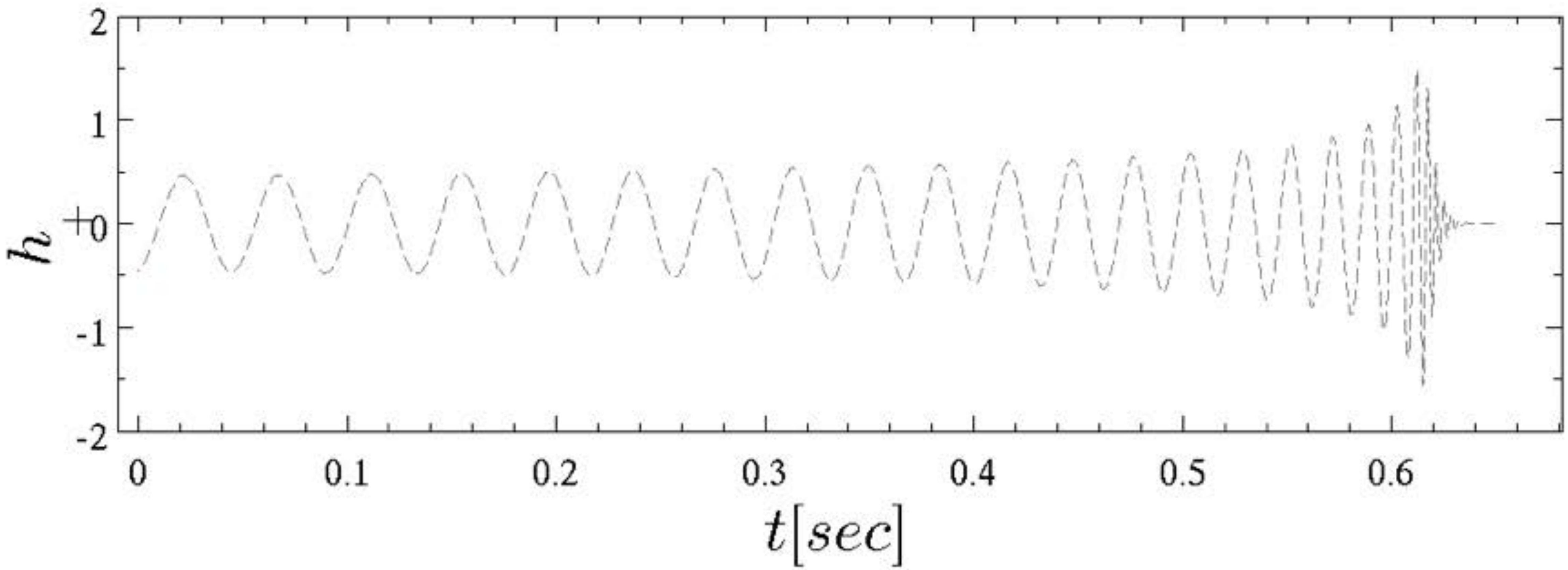
- The system must radiate angular momentum
- How?Use PN-based (Taylor-expanded) radiation reaction force (ang-mom flux)
- Need flux resummation

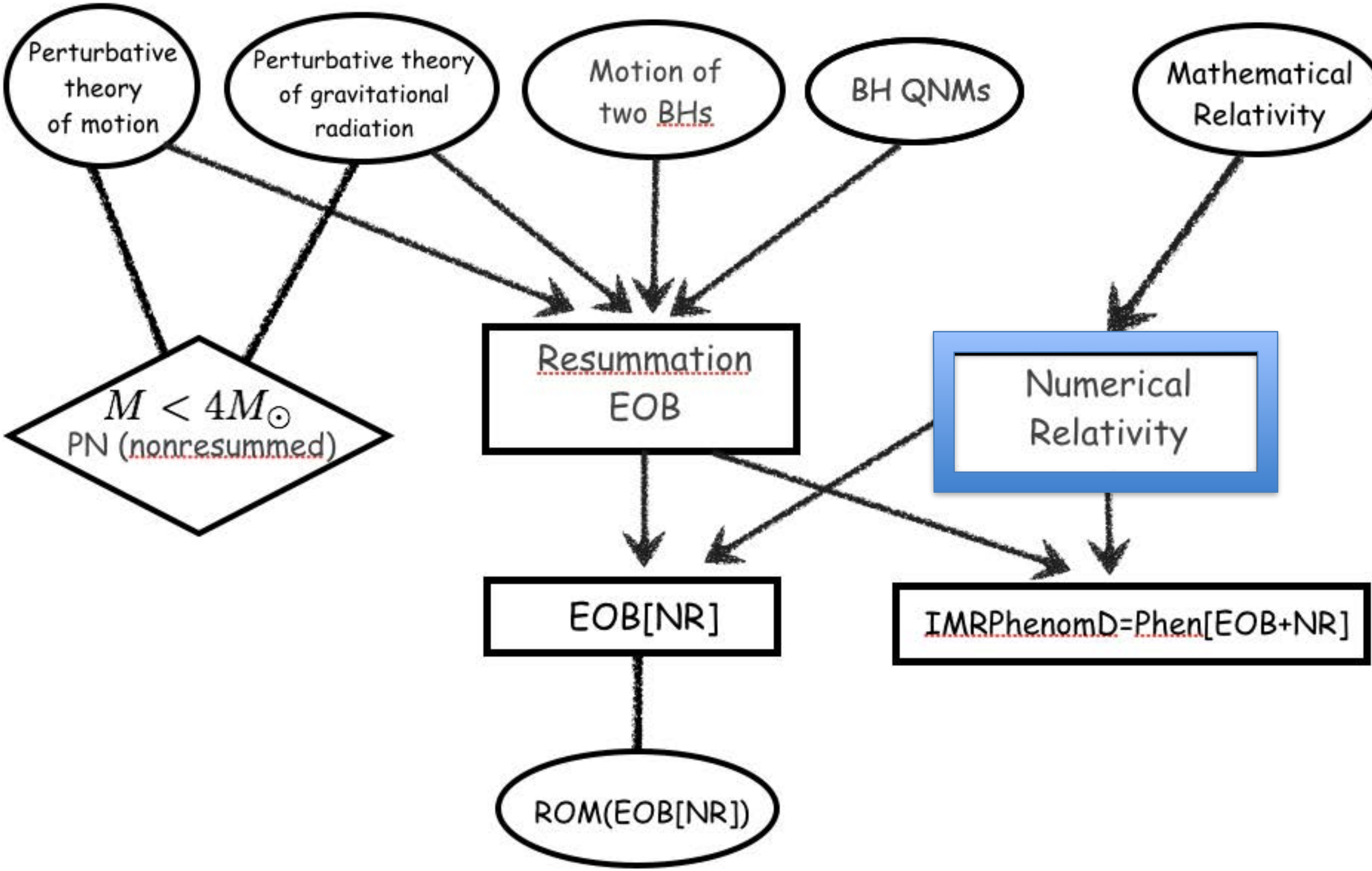
$$\hat{\mathcal{F}}_{\varphi}^{\rm Taylor} = -\frac{32}{5}\nu\Omega^5 r_\Omega^4 \hat{F}^{\rm Taylor}(v_\varphi)$$

Plus horizon contribution [Nagar&Akcay2012]

Resummation multipole by multipole (Damour&Nagar 2007, Damour, Iyer & Nagar 2008, Damour & Nagar, 2009)




2
1
0
-1
-2
0
0.1
0.2
0.3
0.4
0.5
0.6
h_+
$t[sec]$



Perturbative theory of motion
Perturbative theory of gravitational radiation
Motion of two BHs
BH QNMs
Mathematical Relativity
$M < 4M_\odot$
PN (nonresummed)
Resummation EOB
Numerical Relativity
EOB[NR]
IMRPhenomD=Phen[EOB+NR]
ROM(EOB[NR])

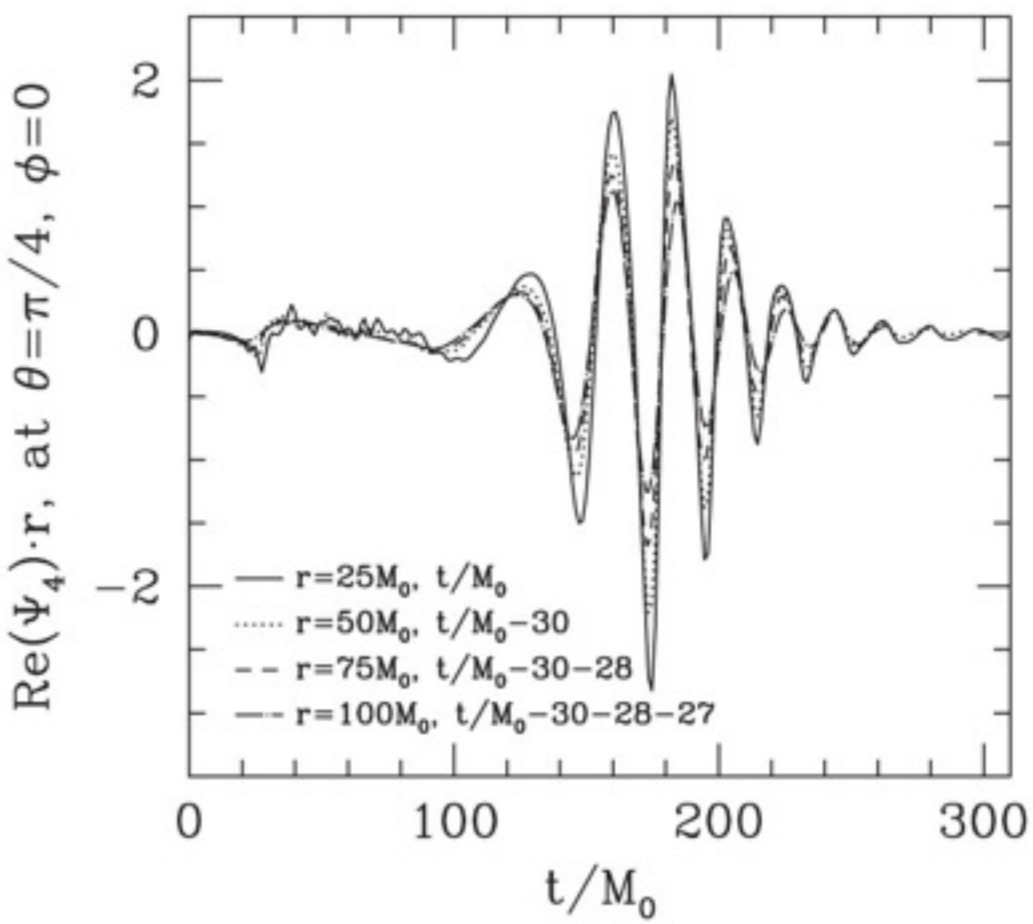
Numerical Relativity (NR)

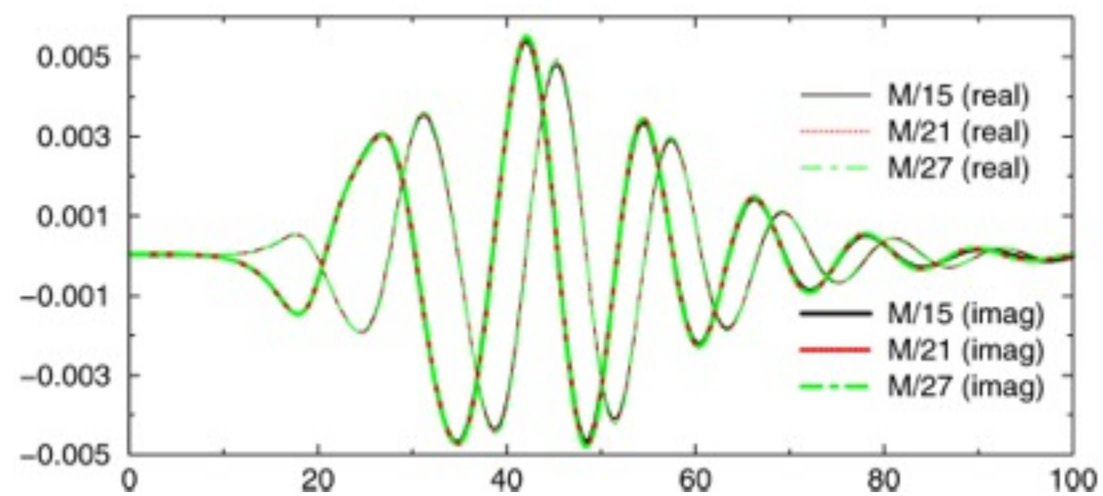
Mathematical foundations : Darmois 27, Lichnerowicz 43, Choquet-Bruhat 52-

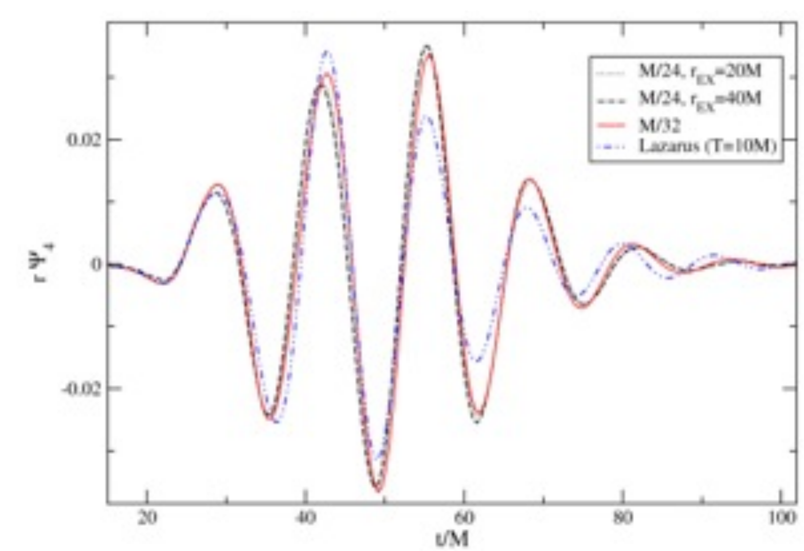
Breakthrough:
Pretorius 2005 generalized harmonic coordinates, constraint damping, excision

Campanelli-Lousto-Maronetti-Zlochover 2006
Baker-Centrella-Choi-Koppitz-van Meter 2006

Moving punctures



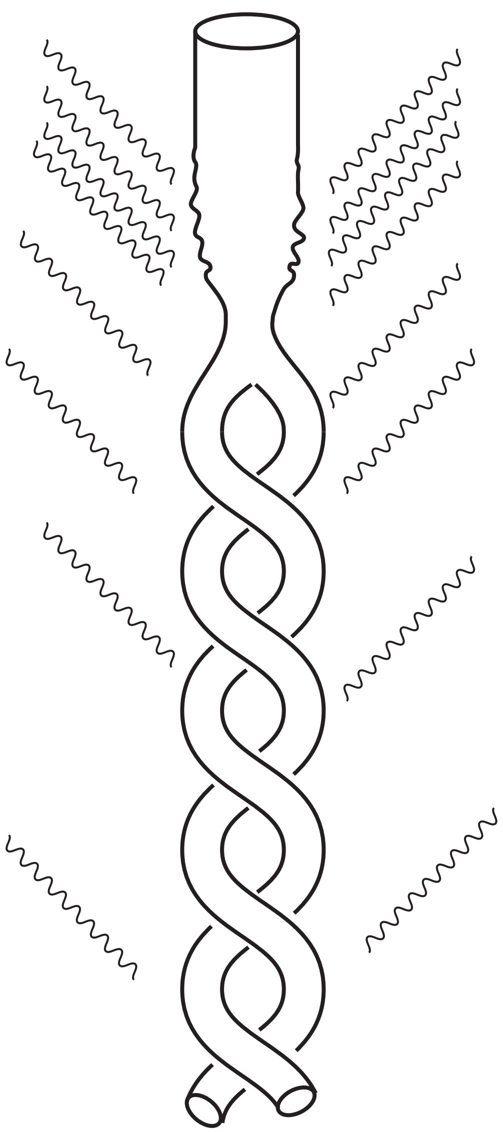




Excision + generalized harmonic coordinates (Friedrich, Garfinkle)

$$C_a \equiv g_{ab}\left(H^a - \Box x^a\right) = 0.$$

+ Constraint damping (Brodbeck et al., Gundlach et al., Pretorius, Lindblom et al.)



$$\begin{aligned} \frac{1}{2} g^{cd} g_{ab,cd} &\ + \\ g^{cd}{}_{(,a} g_{b)d,c} + H_{(a,b)} &\ - \ H_d \Gamma^d_{ab} + \Gamma^c_{bd} \Gamma^d_{ac} \\ +\kappa \left[n_{(a} C_{b)} \right. &\ - \ \left. \frac{1}{2} g_{ab} n^d C_d \right] \\ &\ = \ -8\pi \left(T_{ab} - \frac{1}{2} g_{ab} T \right). \end{aligned}$$

$$\Box C^a = -R^a{}_b C^b + 2\kappa \nabla_b \left[n^{(b} C^{a)} \right],$$

The first EOB vs NR comparison

Buonanno-Cook-Pretorius 2007

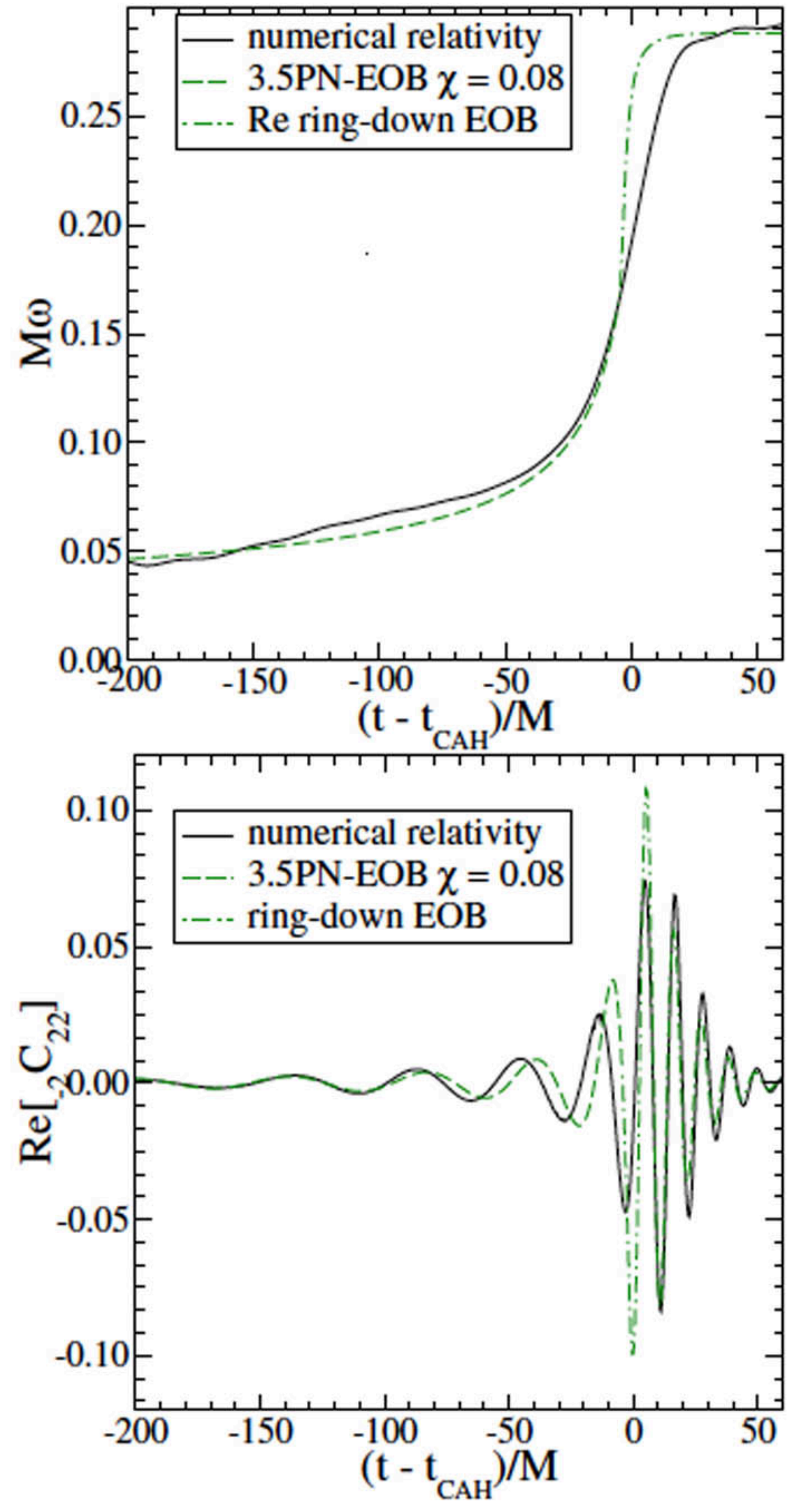
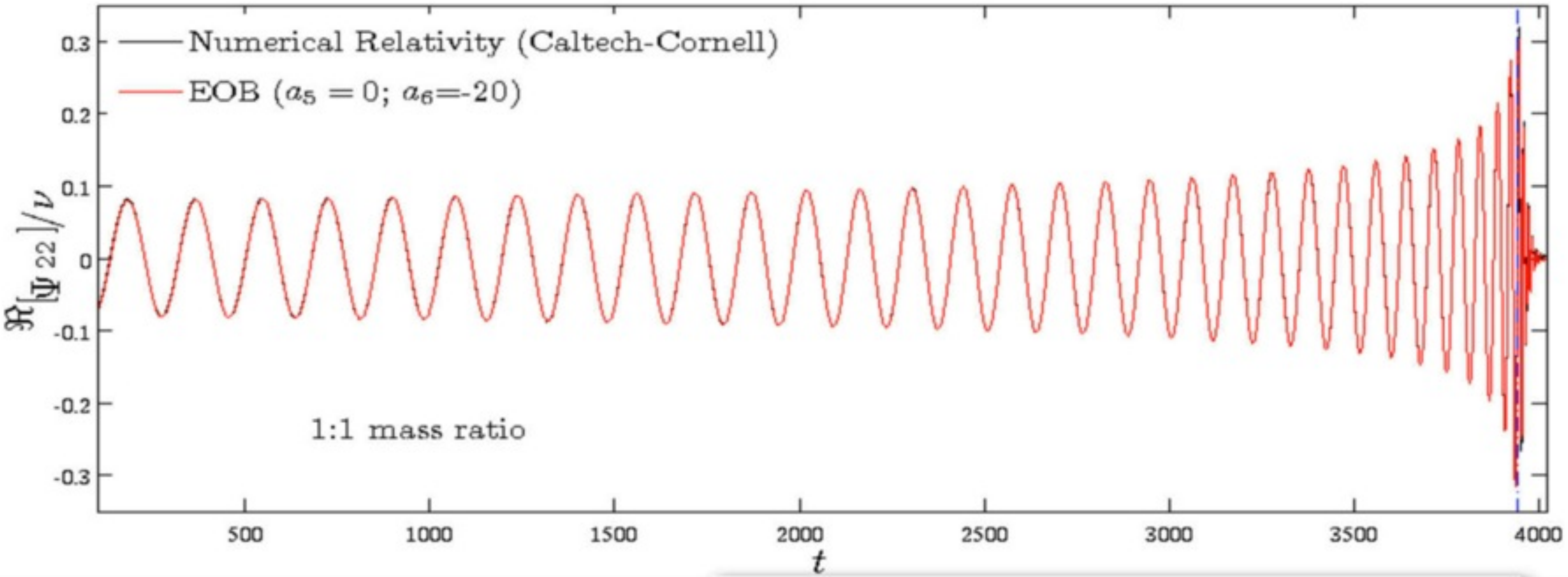
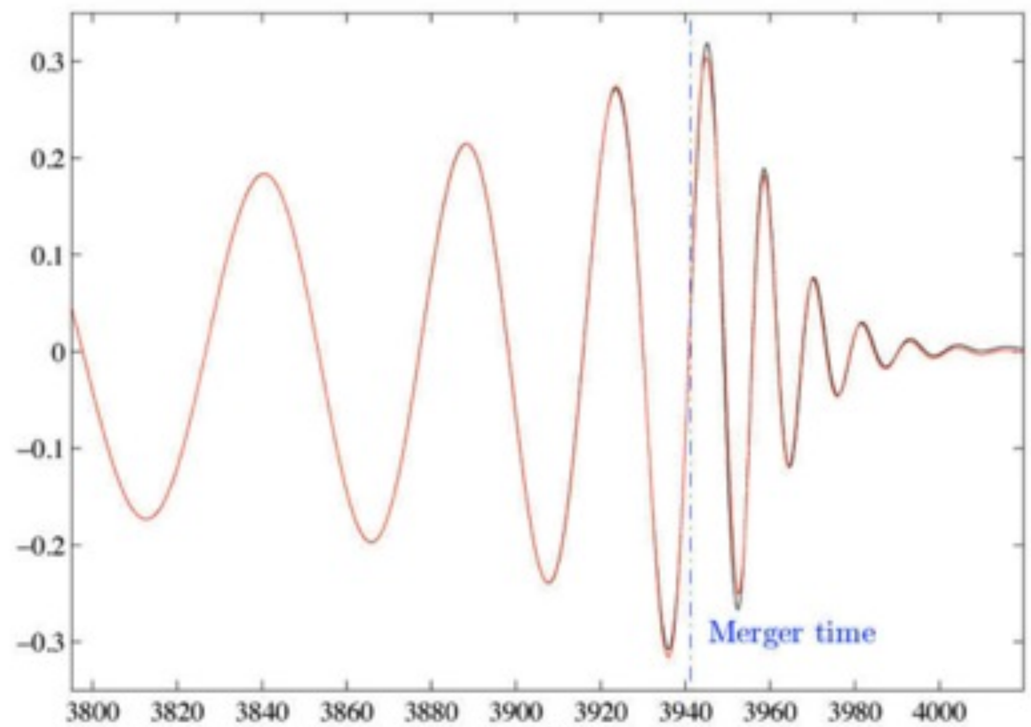
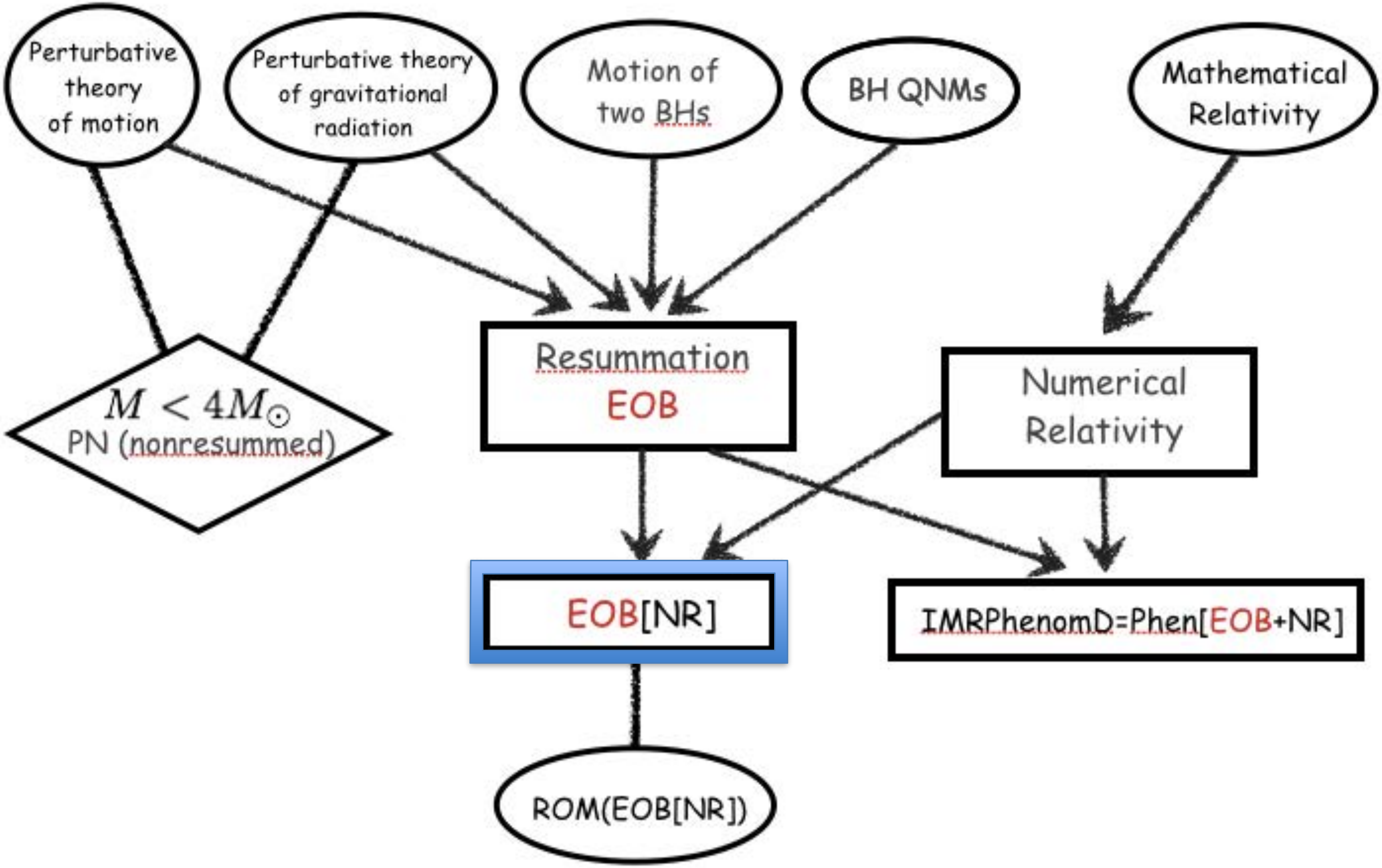


FIG. 21 (color online). We compare the NR and EOB frequency and $\mathrm{Re}[{}_{-2}C_{22}]$ waveforms throughout the entire inspiral–merger–ring-down evolution. The data refers to the $d = 16$ run.

Numerical Relativity Waveform (Caltech-Cornell, SXS)







EOB[NR]: Damour-Gourgoulhon-Grandclement '02, Damour-Nagar '07-16, Buonnano-Pan-Taracchini-….'07-16

NR-completed resummed 5PN EOB radial A potential

4PN analytically complete + 5 PN logarithmic term in the A(u, nu) function,
With u = GM/R and nu = m1 m2 / (m1 + m2)^2

[Damour 09, Blanchet et al 10, Barack-Damour-Sago 10, Le Tiec et al 11, Barausse et al 11, Akcay et al 12, Bini-Damour 13, Damour-Jaranowski-Schäfer 14, Nagar-Damour-Reisswig-Pollney 15]

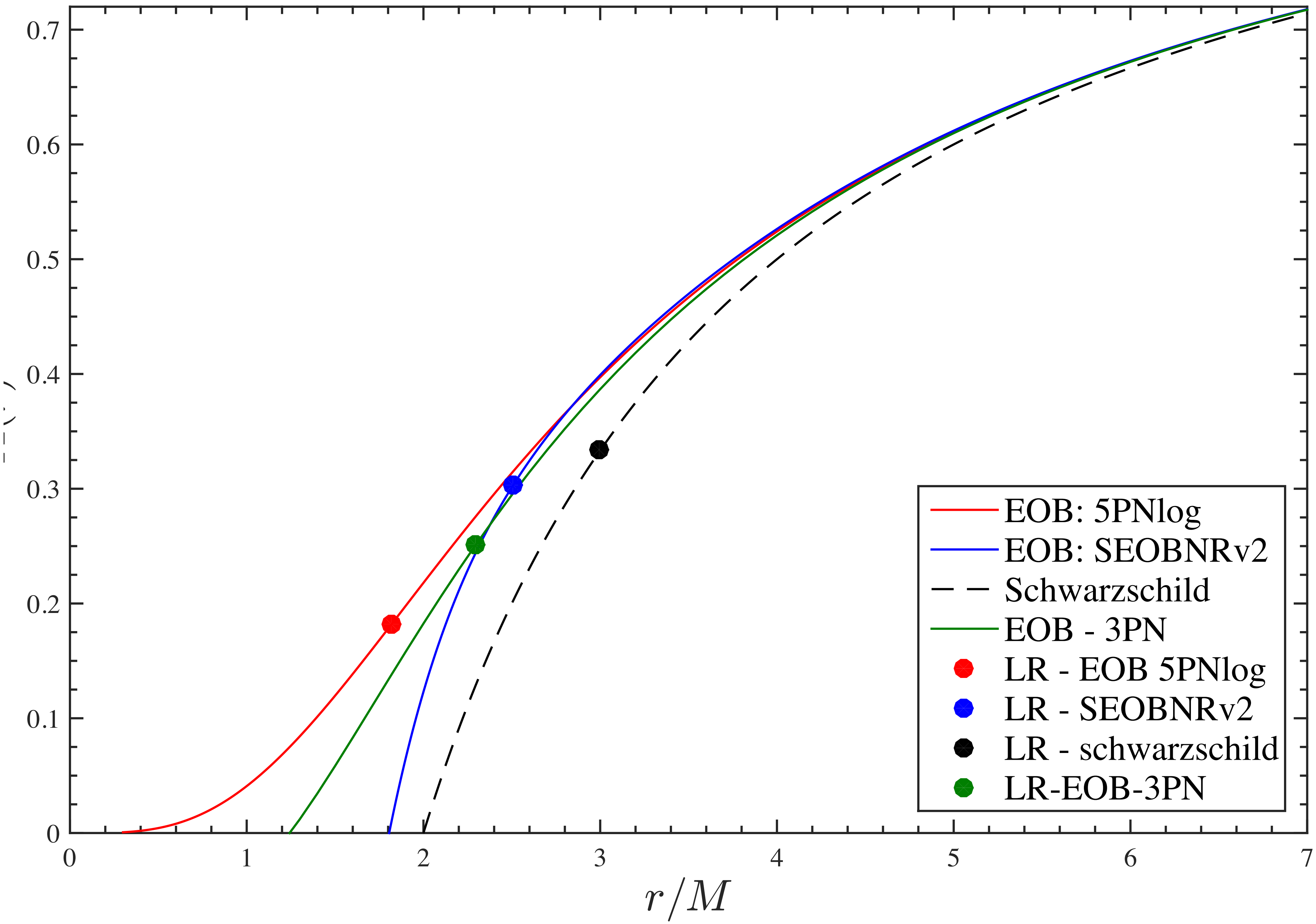
$$
\begin{aligned}
A(u;\nu,a_6^c) \;=\; & P_5^1\Bigg[1-2u+2\nu\,u^3+\nu\left(\frac{94}{3}-\frac{41}{32}\pi^2\right)u^4 \\
& +\;\nu\left[-\frac{4237}{60}+\frac{2275}{512}\pi^2+\left(-\frac{221}{6}+\frac{41}{32}\pi^2\right)\nu+\frac{64}{5}\ln(16e^{2\gamma}u)\right]u^5 \\
& +\;\nu\left[a_6^c(\nu)-\left(\frac{7004}{105}+\frac{144}{5}\nu\right)\ln u\right]u^6\Bigg]
\end{aligned}
$$

$$
a_6^{c\,\text{NR-tuned}}(\nu) = 81.38 - 1330.6\,\nu + 3097.3\,\nu^2
$$

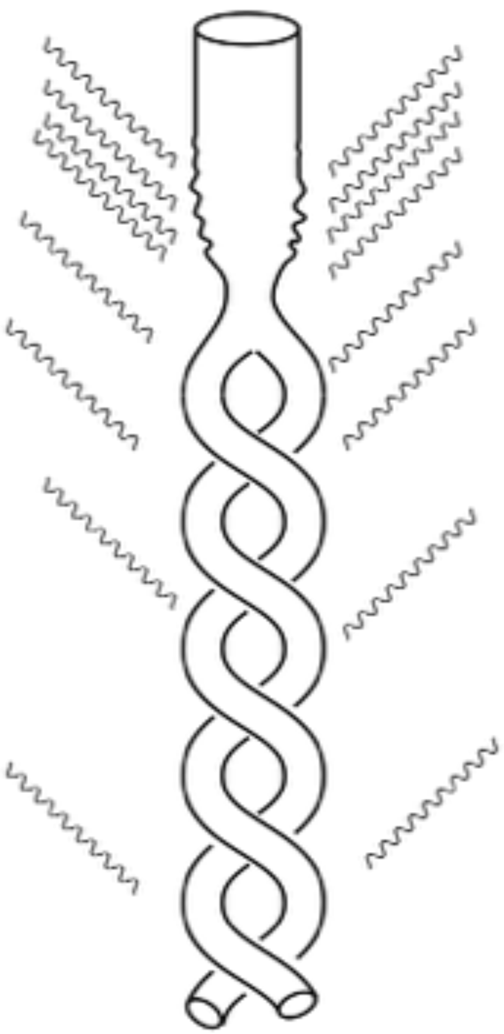
MAIN RADIAL RADIAL EOB POTENTIAL A(R)

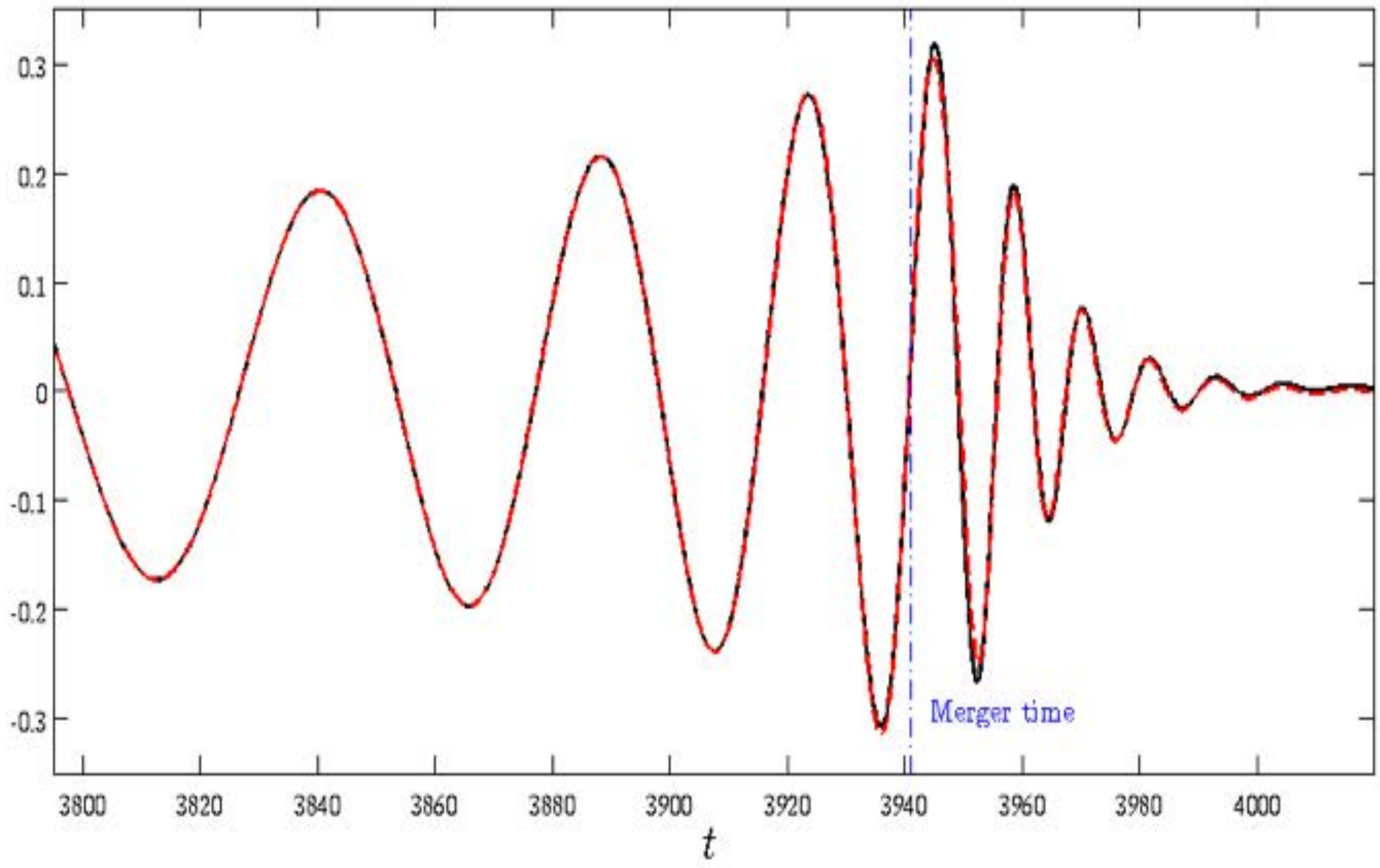
m1=m2 case

A(r)

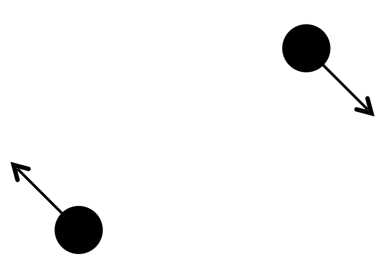


EOB / NR Comparison



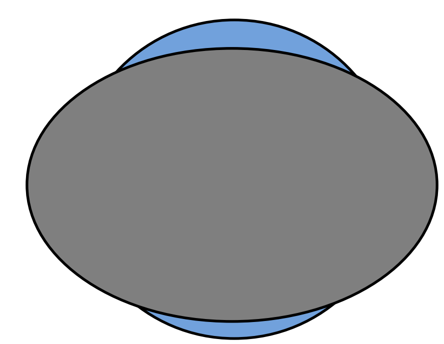


Inspiral + « plunge »



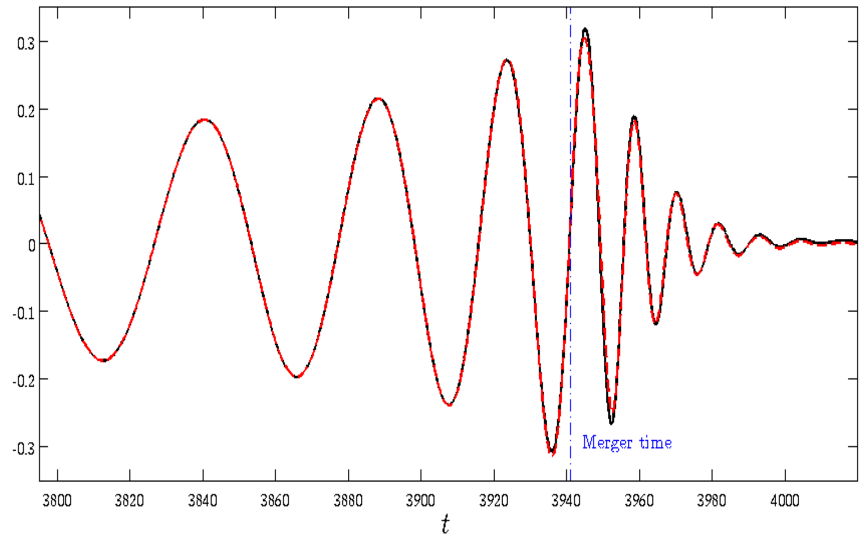
Two orbiting point-masses:
Resummed dynamics

Ringing BH

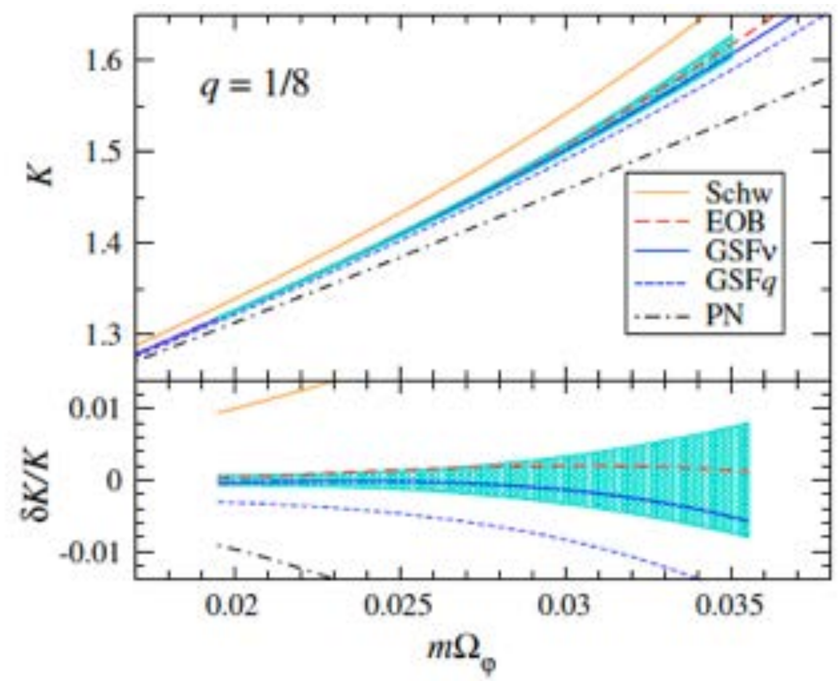


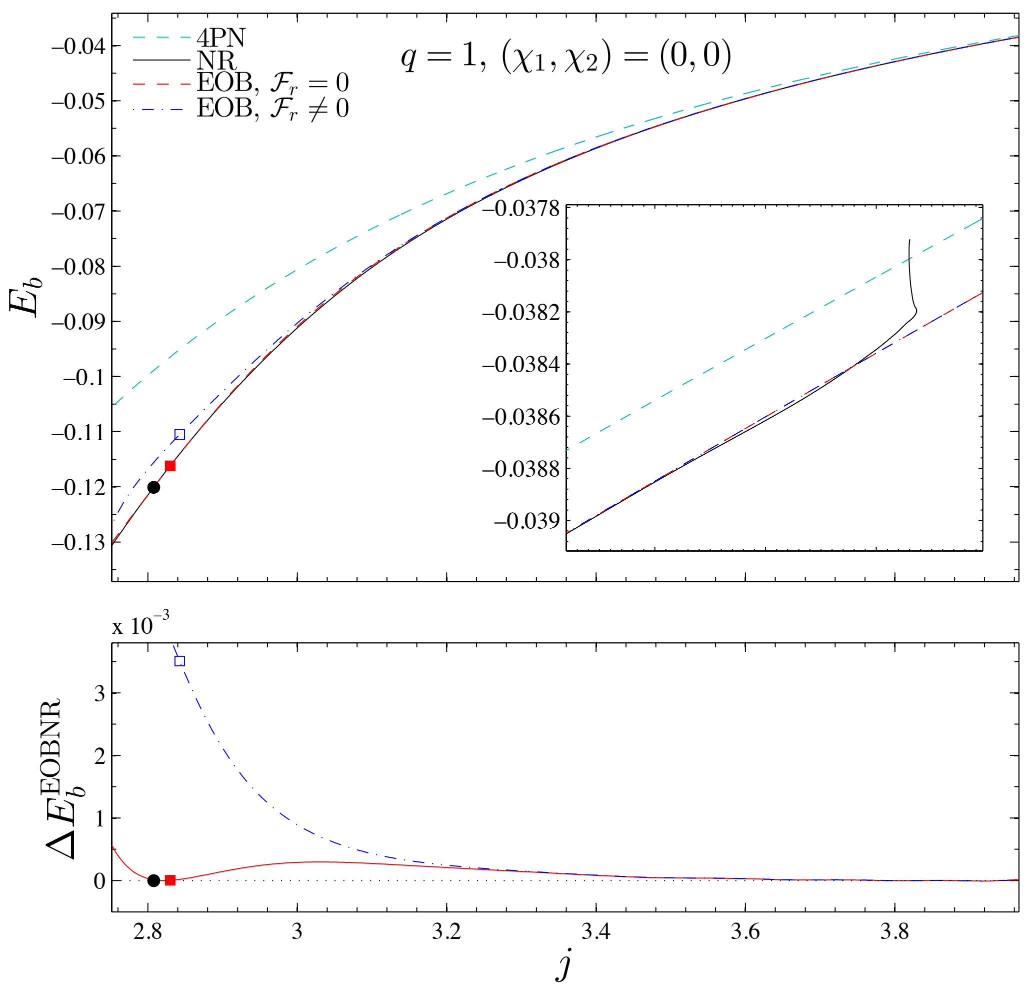
Peak emitted power ~ 3 x 10^56 erg/s ~ 0.001 c^5/G

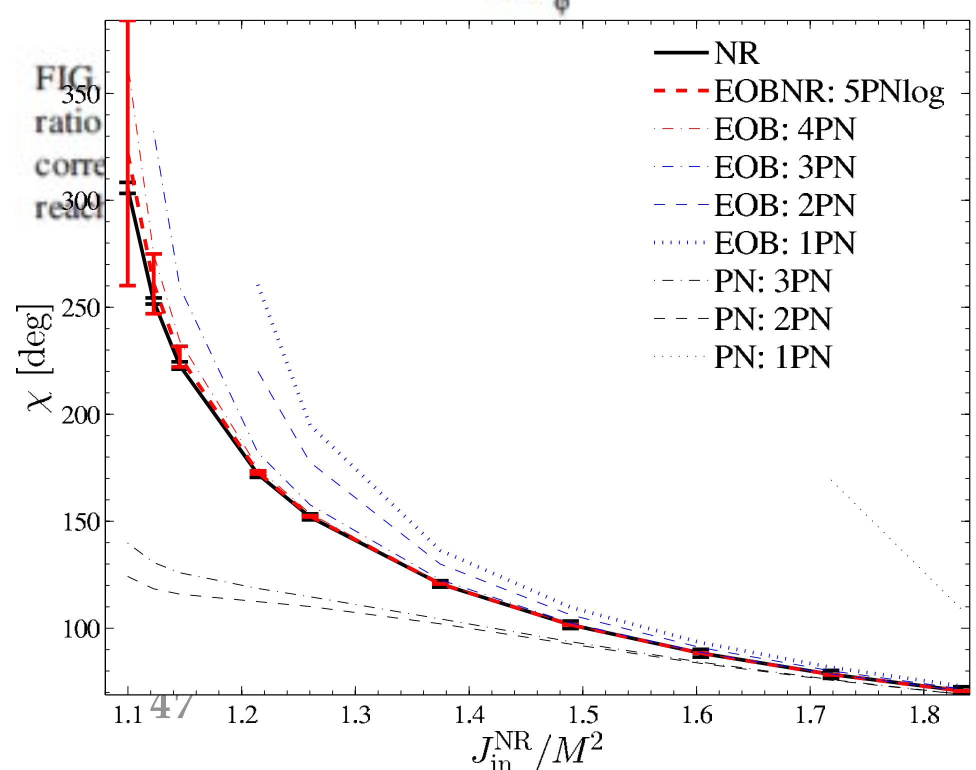
EOB VS NR



waveform (Damour-Nagar 09),
energetics(Nagar-Damour-Resswig-Pollney 16),
periastron precession (LeTiec-Mroue-Barack-Buonanno-Pfeiffer-Sago-Tarachini 11, and
scattering angle (Damour-Guercilena-Hinder-Hopper-Nagar-Rezzolla 14)





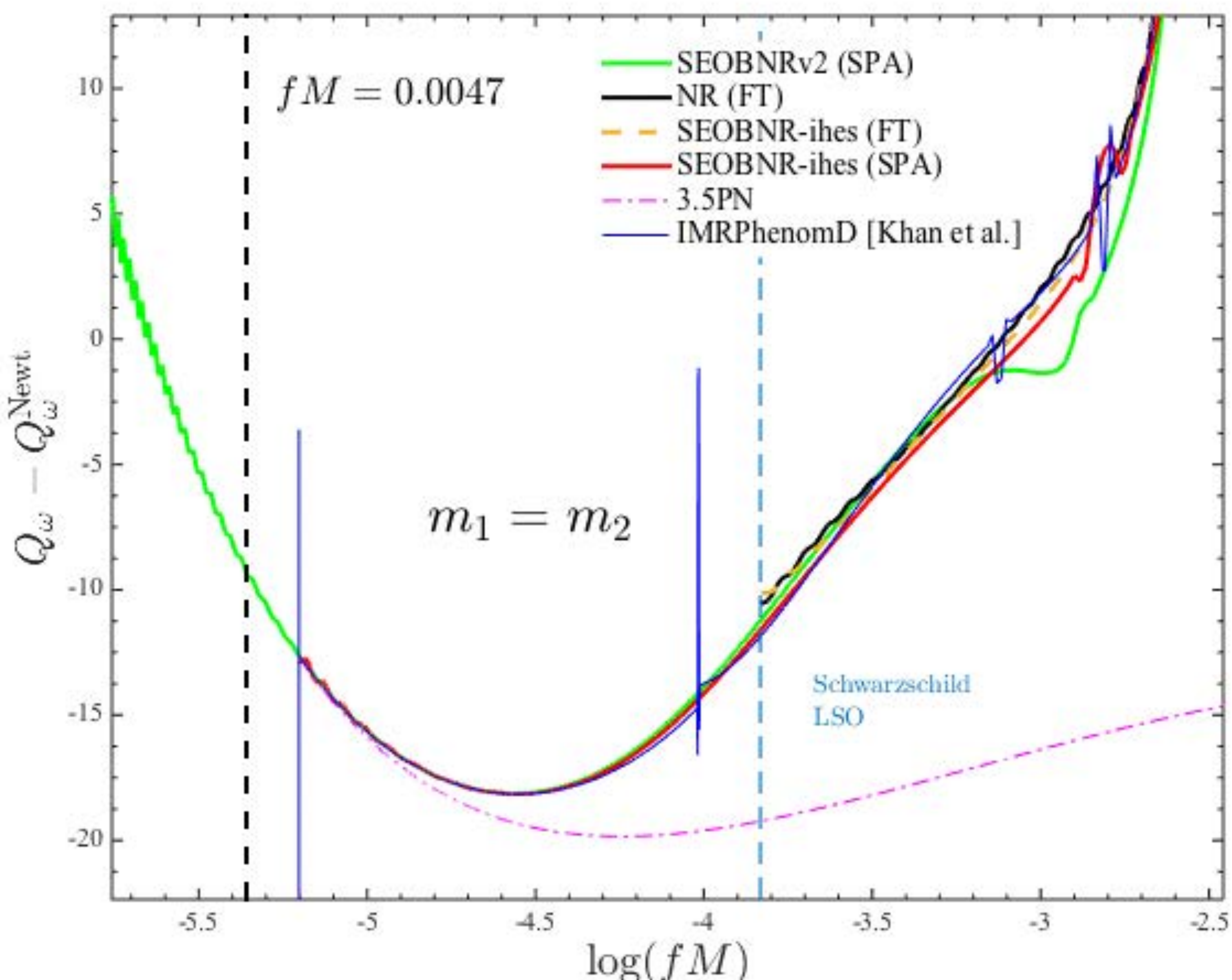


PN, EOB, NR, PHENOMD

PN accuracy loss during inspiral

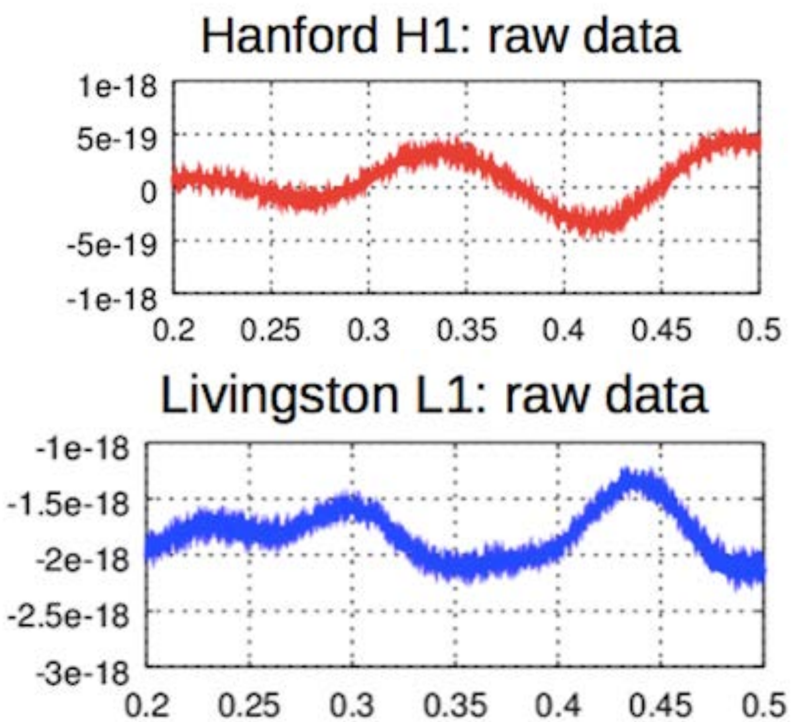
Dimensionless « quality factor » of GW phase $Q_\omega = f^2 \frac{d^2\psi(f)}{df^2} \approx \frac{\omega^2}{\dot{\omega}}$

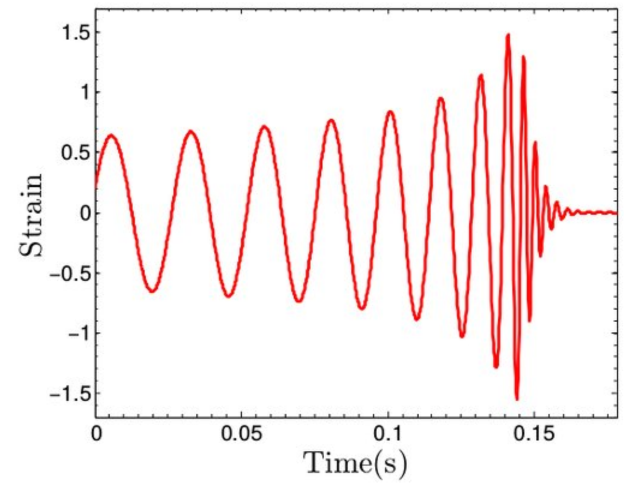
$Q_\omega - Q_\omega^N$



GW150914 and GW151226: incredibly small signals lost in the broad-band noise

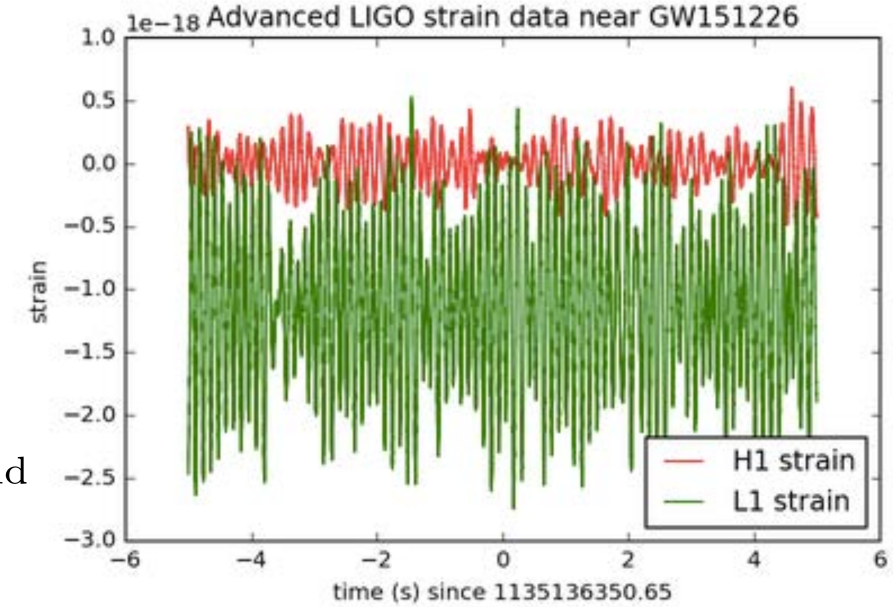
GW150914, from Chassande-Mottin 5 April 2016

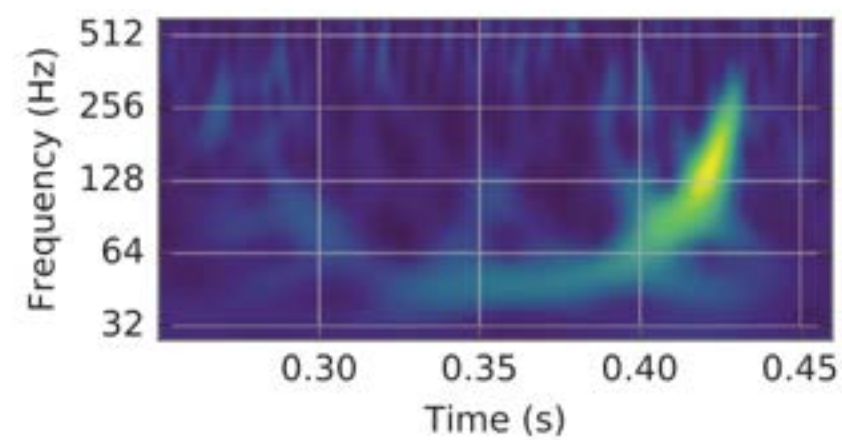




$$h_{GW}^{\max} \sim 10^{-21} \sim 10^{-3}\, h_{LIGO}^{\text{broadband}}$$

GW151226 from LIGO open data





Several levels of signal search and analysis:

1. online time-frequency analysis (Wilson, Meyer, Daubechies-Jaffard-Journe, Klimenko et al.)
2. online matched-filter analysis (EOB[NR] and Phenom[EOB +NR])
3. offline matched-filter analysis (significance, parameter estimation)(EOB[NR] templates with PyCBC or GstLAL)

Matched Filtering

$$\langle output | h_{\text{template}} \rangle = \int \frac{df}{S_n(f)} o(f) h^*_{\text{template}}(f)$$

MATCHED FILTERING SEARCH AND DATA ANALYSIS

Precomputed bank of 250 000 templates for inspiralling and coalescing BBH GW waveforms: m1, m2, chi1=S1/m1^2, chi2=S2/m2^2

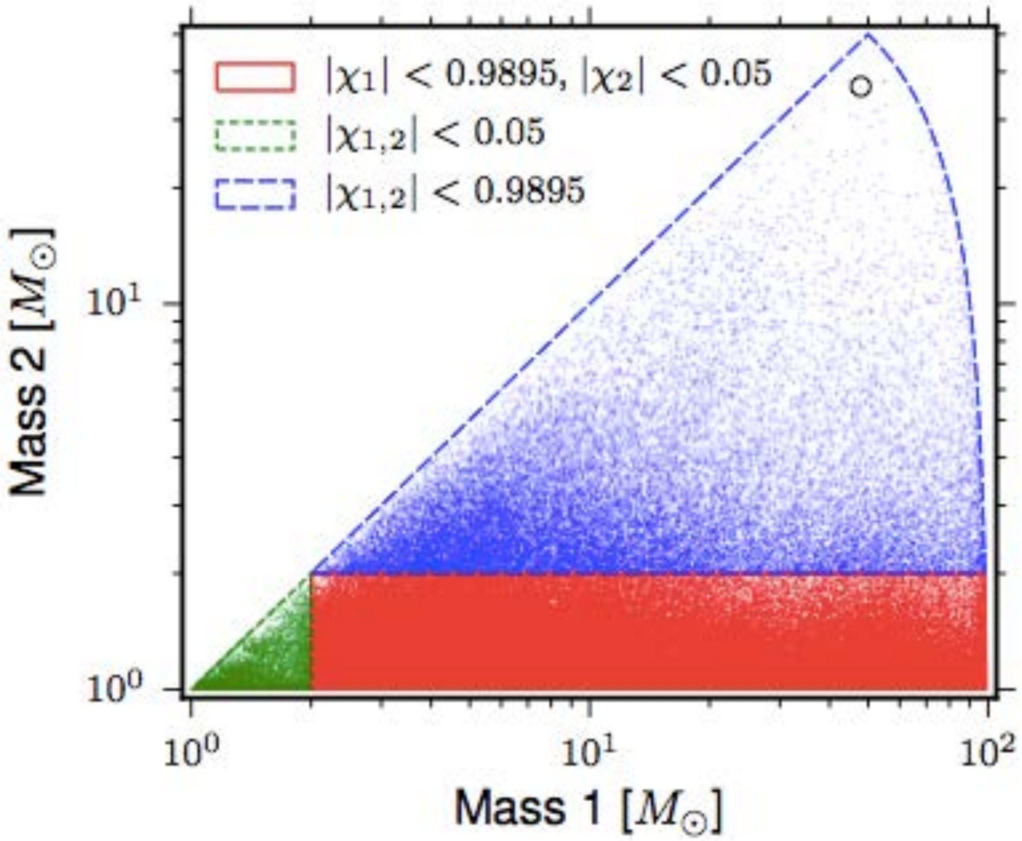
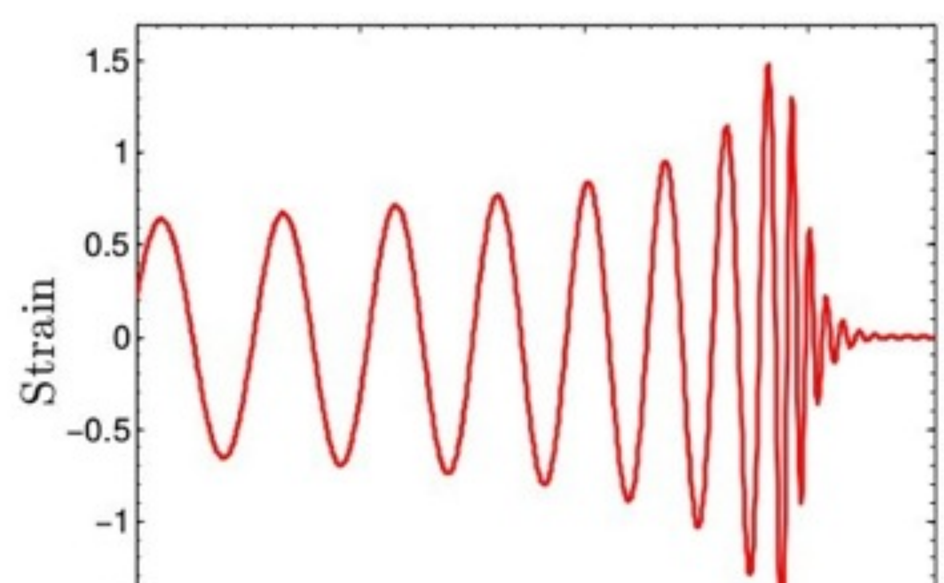
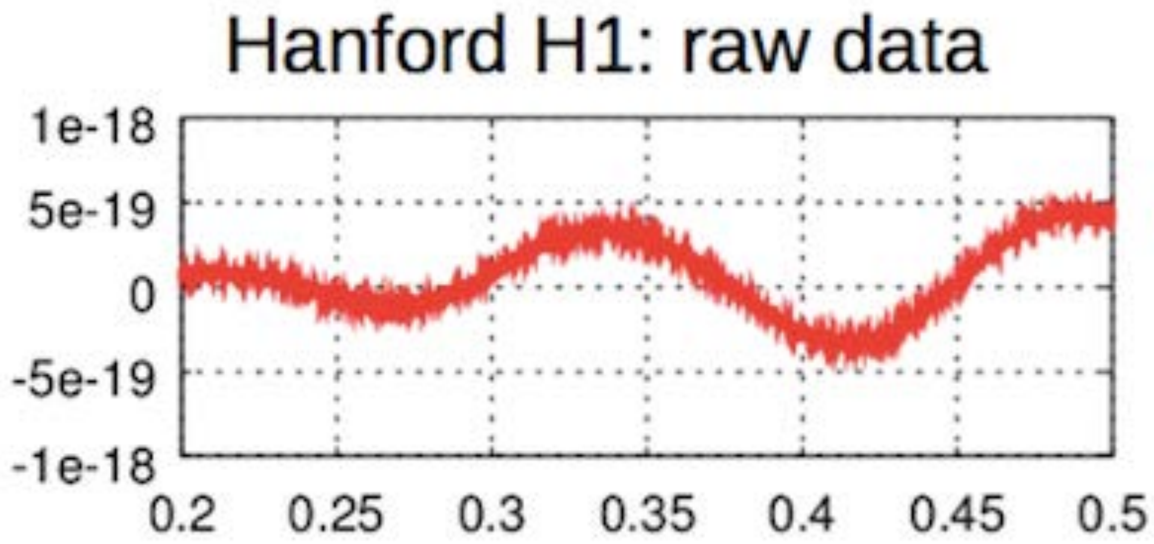


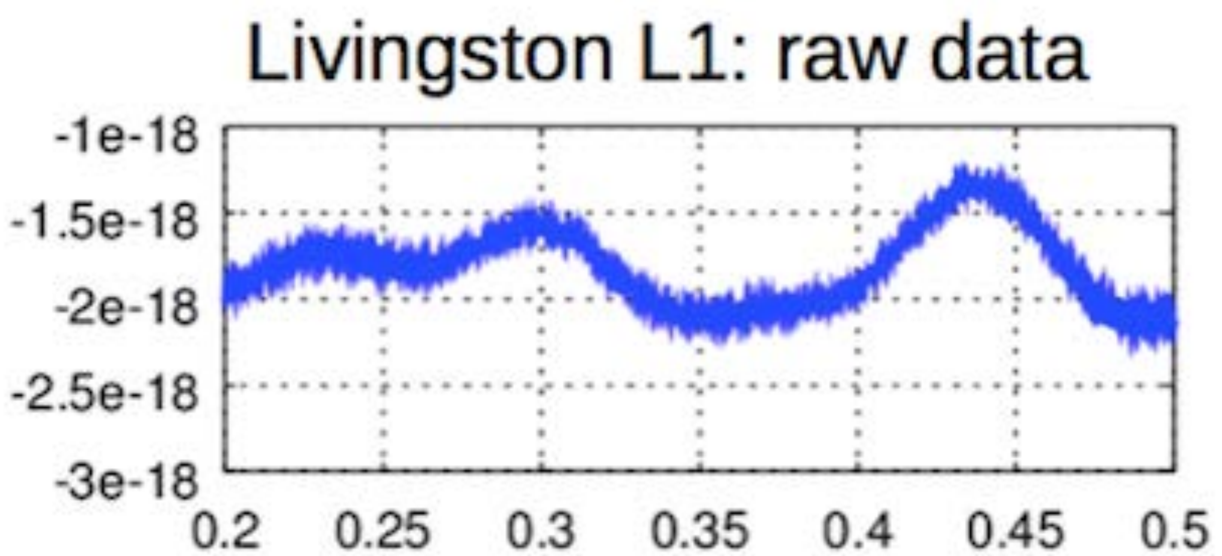
FIG. 1. The four-dimensional search parameter space covered by the template bank shown projected into the component-mass plane, using the convention $m_1 > m_2$. The lines bound mass regions with different limits on the dimensionless aligned-spin parameters χ_1 and χ_2. Each point indicates the position of a template in the bank. The circle highlights the template that best matches GW150914. This does not coincide with the best-fit parameters due to the discrete nature of the template bank.

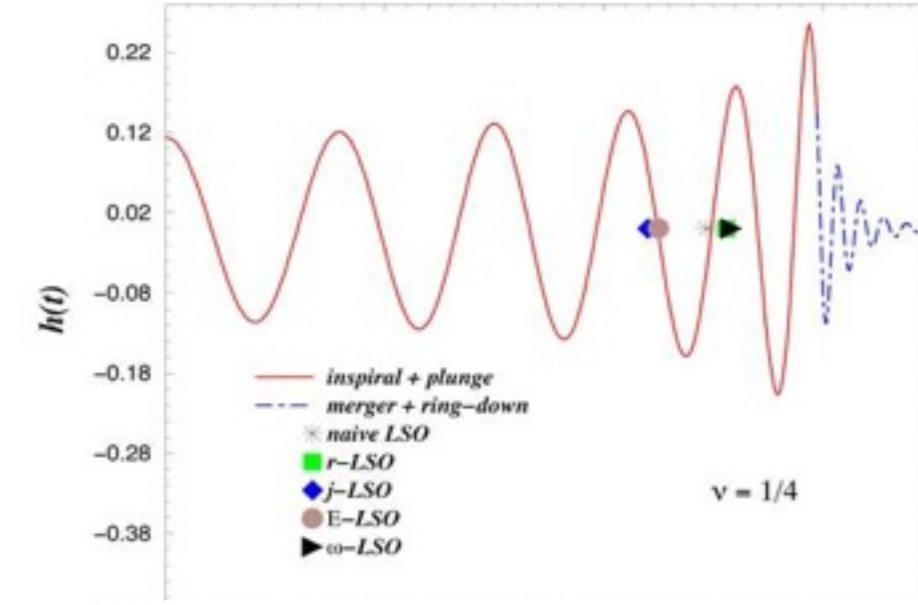
Search template bank made of SpinningEOB[NR] templates (Buonanno-Damour 99, Damour '01…,Taracchini et al. 14) in ROM form; Recently improved (Bohé et al '16') by including leading 4PN terms (Bini-Damour '13), spin-dependent terms (Pan-Buonnano et al. '13), and calibrating against 141 NR simulations.



GW150914 vs EOB[NR]

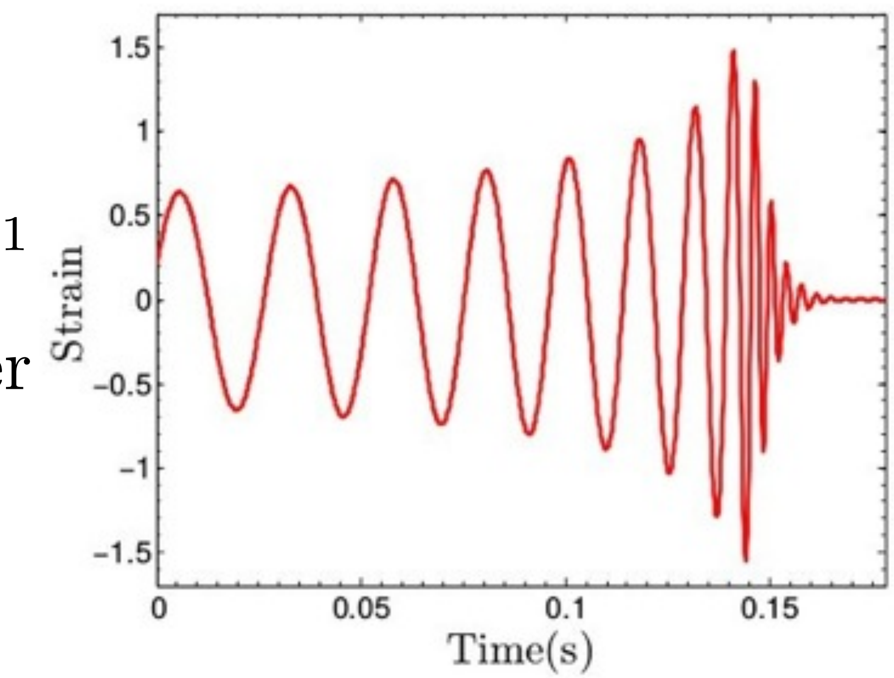




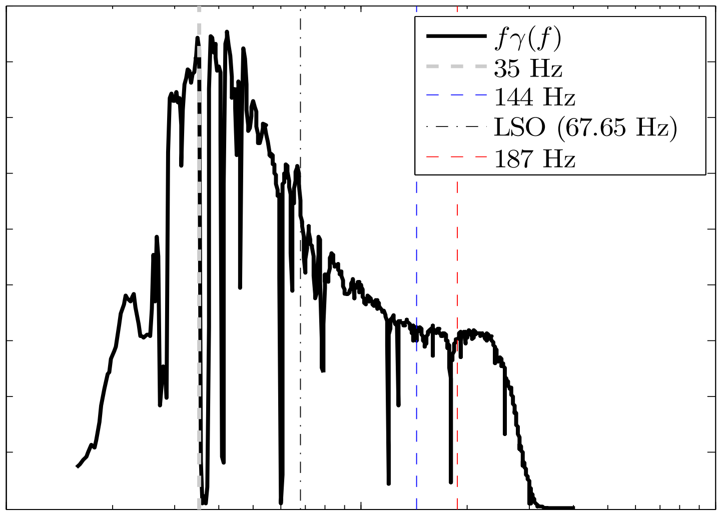


scale : 10^{-21}

$500 \times$ smaller



$$\frac{d\rho^2}{d\ln f} = \frac{f|\tilde{h}(f)|^2}{S_n(f)}$$



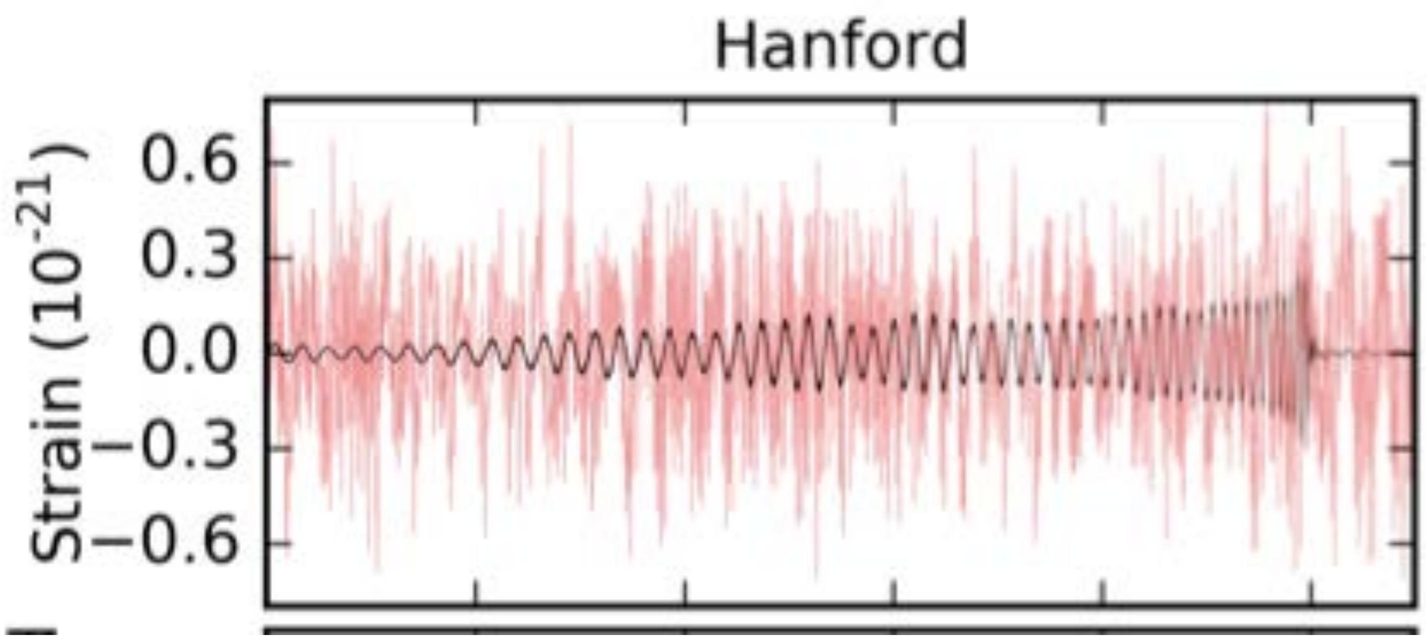
$$m_1 = 36^{+5}_{-4} M_\odot$$

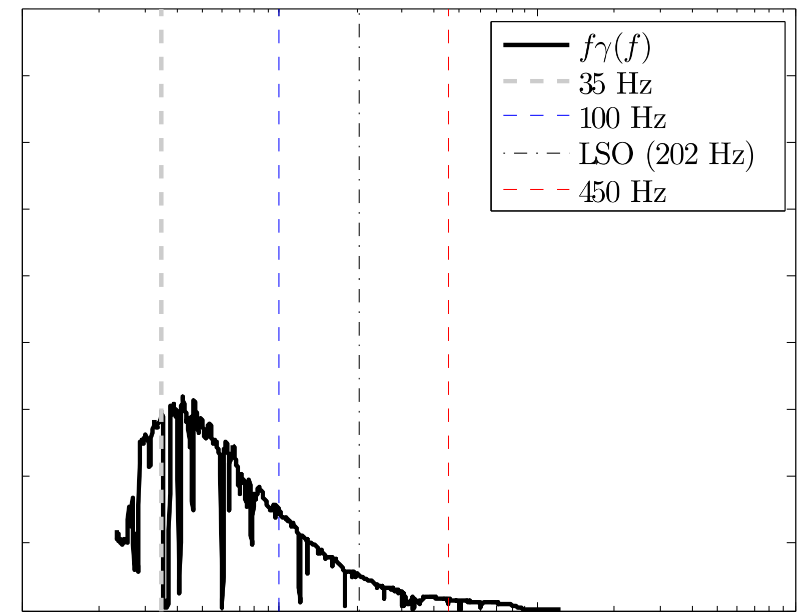
$$m_2 = 29^{+4}_{-4} M_\odot$$

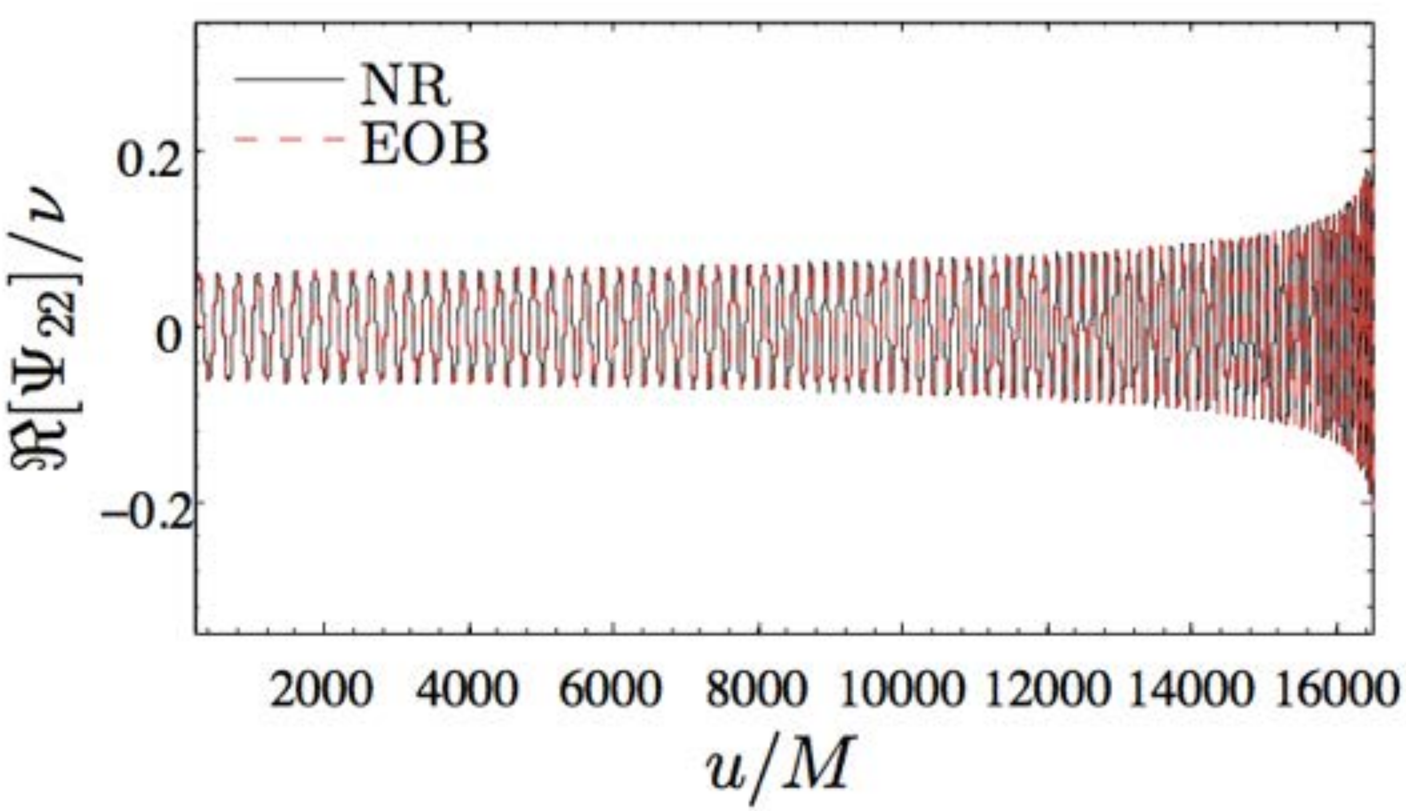
$$\chi_{\rm eff} = -0.06^{+0.17}_{-0.18}$$

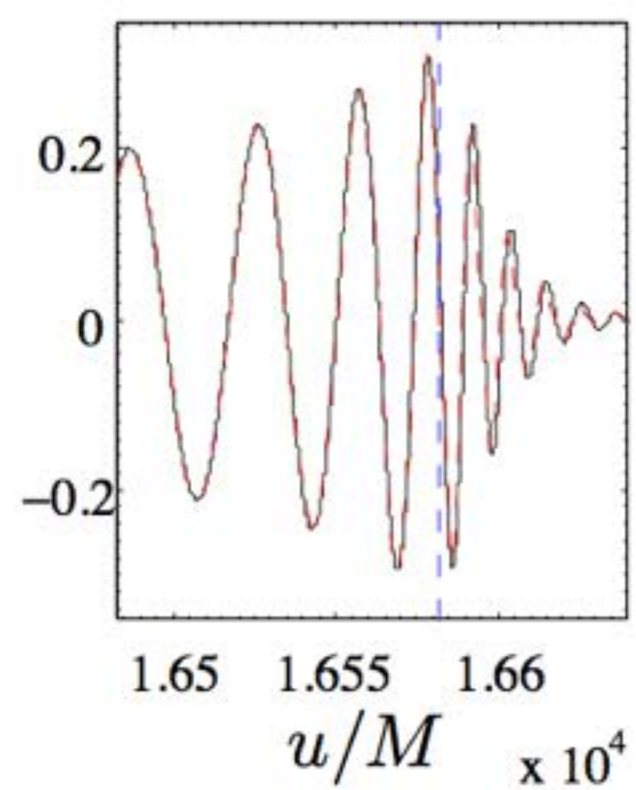
$$D_{\rm L} = 410^{+160}_{-180} {\rm Mpc}$$

GW151226: only detected via accurate matched filters









$$m_1 = 14.2^{+8.3}_{-3.7} M_\odot$$

$$m_2 = 7.5^{+2.3}_{-2.3} M_\odot$$

$$\chi_{\rm eff} = +0.21^{+0.20}_{-0.10}$$

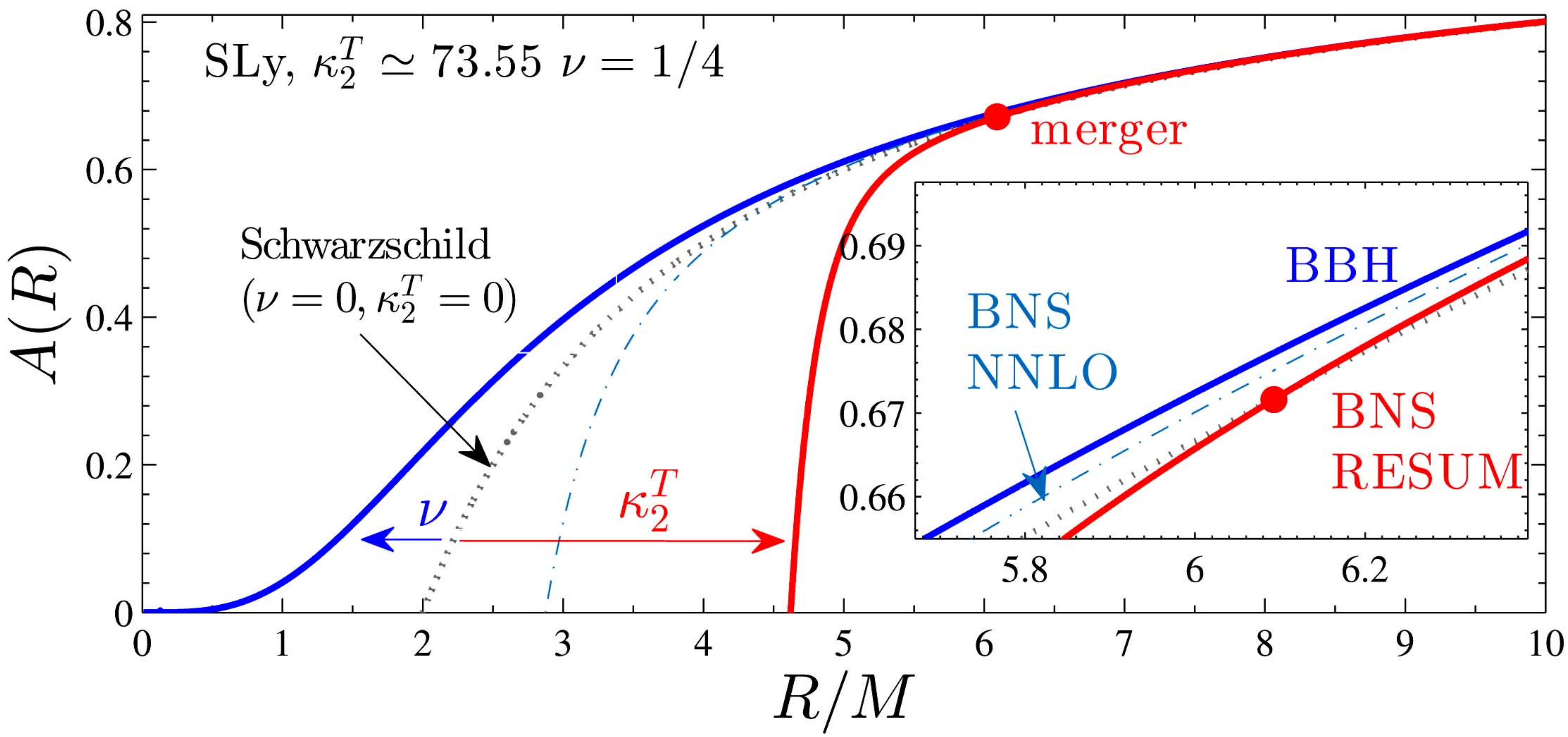
$$D_{\rm L} = 440^{+180}_{-190} {\rm Mpc}$$

FUTURE PROSPECT 1: NSNS AND BHNS GW: PROBING THE NUCLEAR EOS FROM LATE INSPIRAL TIDAL EFFECTS IN NSNS OR BHNS

Tidal extension of EOB (TEOB) [Damour-Nagar 09]

$$A(r) = A_r^0 + A^{\rm tidal}(r)$$
$$A^{\rm tidal}(r) = -\kappa_2^T u^6 \left(1 + \bar{\alpha}_1 u + \bar{\alpha}_2 u^2 + \ldots\right) + \ldots$$

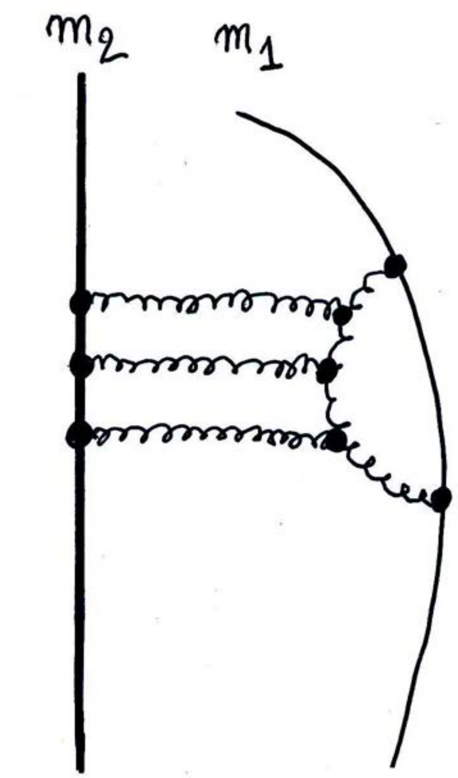
TEOB[NR] A(R) potential (Bernuzzi et al. 2015)



FUTURE PROSPECT 2:
GW DETECTORS IN SPACE LISA

Here also analytical methods (completed by numerical ones) will be important: Gravitational Self-Force program: m1 << m2

- Analytical high-PN results : Blanchet-Detweiler-LeTiec-Whiting ’10, Damour ’10, Blanchet et al ’10, LeTiec et al ’12, Bini-Damour ’13-15, Kavanagh-Ottewill-Wardell ’15
- (gauge-invariant) Numerical results : Detweiler ’08, Barack-Sago ’09, Blanchet-Detweiler-LeTiec-Whiting ’10, Barack-Damour-Sago ’10, Shah-Friedman-Keidl ’12, Dolan et al ’14, Nolan et al ’15, …
- Analytical PN results from high-precision (hundreds to thousands of digits !) numerical results : Shah-Friedman-Whiting ’14, Johnson-McDaniel-Shah-Whiting ’15



EOB[SF] program: import high PN-SF results in EOB (Bini-Damour ’15, Kavanagh et al ’15, Akcay, van de Meent, Hopper, ……)

$$a_{10}^{c} = \frac{186054788842060273}{7079830758000}\ln(2) - \frac{1619008}{405}\zeta(3) - \frac{213398732147280\,97}{1011404394000}\gamma + \frac{27101981341}{100663296}\pi^6 - \frac{6236861670873}{125565440}\ln(3) + \frac{360126}{49}\ln(2)\ln(3) + \frac{180063}{49}\ln(3)^2 - \frac{121494974752}{9823275}\ln(2)^2 - \frac{24229836023352153}{549755813888}\pi^4 + \frac{1115369140625}{124540416}\ln(5) + \frac{96889010407}{277992000}\ln(7) + \frac{7543701437062331862329 9}{186907532011200 00} - \frac{60648244288}{9823275}\ln(2)\gamma + \frac{200706848}{280665}\gamma^2 + \frac{11980569677139}{2306867200}\pi^2 + \frac{360126}{49}\gamma\ln(3),$$

$$a_{10}^{\ln} = -\frac{212751433335120 97}{2022808788000} + \frac{200706848}{280665}\gamma - \frac{30324122144}{9823275}\ln(2) + \frac{180063}{49}\ln(3),$$

$$a_{10}^{\ln^2} = \frac{50176712}{280665},$$

Conclusions

• One century after the work of Einstein (and Schwarzschild), thanks to the LIGO observations (+ the LIGO-Virgo analysis):
1.We have seen vibrations of the space geometry pass by the Earth, and
2.We have got the first (almost) direct evidence for the existence of BHs.

• Several aspects of Analytical Relativity have played a key role: perturbative theory of motion, perturbative theory of GW generation, EOB formalism. The analytical EOB method had predicted in 2000 the complete GW signal emitted by the coalescence of two black holes. This was confirmed, and refined, in 2005 by Numerical Relativity. The union between Analytical Relativity and Numerical Relativity (particularly EOB+NR) has been crucial for computing the 250, 000 theoretical GW templates $h(t;m_1, m_2, S_1, S_2)$ which have been used for extracting the GW signals from the noise by matched filtering, for assessing their physical significance, and for measuring the source parameters. One expects most of the BBH (and BNS) signals to be detected only by means of accurate EOB[NR] templates (as was the case for GW151226).

• The detailed study of BBH coalescence signals opens a new window for probing relativistic gravity: v/c =1/2; GM/(c^2 r) =1/2 . We hope to have soon indirect confirmations of the GR-predicted physical properties of BH (QNMs …), and of the (TT) structure of GWs.

• Opening of a new window on the universe: GW astronomy: might be dominated by BBH (Belczynski et al 2010); waiting for BNS + EM signal (GRB ?), and for LIGO/Virgo/Kagra/Indigo network. The detailed study of coalescences involving NS will open a window on the EOS of nuclear matter (tidal polarizability).

• Window on cosmic-size strings especially via GW bursts above the Gaussian background
• Potentially new window on early universe via GW cosmological background
• Future GW detectors: space detectors, second generation ground-based detectors,…